Dynamic Asset Allocation with Predictable Returns and Transaction Costs

Online Appendices

A General quadratic objective function

It is straight-forward to extend our approach to a non-zero risk-free rate $R_{0,t}$ and an objective function that is linear-quadratic in the position vector (i.e., $F(x_t, w_T) = w_T + a_1^\top x_T - \frac{1}{2}x_T^\top a_2\, x_T$) rather than linear in total wealth. The $F(\cdot,\cdot)$ function parameters could be chosen to capture different objectives, such as maximizing the terminal gross value of the position ($w_T + \mathbf{1}^\top x_T$) or the terminal liquidation (i.e., net of transaction costs) value of the portfolio ($w_T + \mathbf{1}^\top x_T - \frac{1}{2}x_T^\top \Lambda_T x_T$), or the terminal wealth penalized for the riskiness of the position ($w_T + \mathbf{1}^\top x_T - \frac{\gamma}{2}x_T^\top \Sigma_T x_T$), or some intermediate situation.

Suppose the objective function is:

$$\max_{u_1,\ldots,u_T} \mathsf{E}\left[F(w_T, x_T) - \sum_{t=0}^{T-1} \frac{\gamma}{2} x_t^\top \Sigma_{t\to t+1} x_t\right] \tag{51}$$

By recursive substitution x_T and w_T can be rewritten as:

$$x_T = x_0 \circ R_{0\to T} + \sum_{t=1}^{T} u_t \circ R_{t\to T} \tag{52}$$

$$w_T = w_0 R_{0,0\to T} - \sum_{t=1}^{T}\left(u_t^\top \mathbf{1} R_{0,t\to T} + \frac{1}{2}u_t^\top \Lambda_t u_t R_{0,t\to T}\right) \tag{53}$$

where we have defined security i's cumulative return between date t and T as:

$$R_{i,t\to T} = \prod_{s=t+1}^{T} R_{i,s} \tag{54}$$

(with the convention that $R_{i,t\to t} = 1$) and the corresponding N-dimensional vector $R_{t\to T} = [R_{1,t\to T}; \ldots; R_{N,t\to T}]$.

Now note that:

$$a_1^\top x_T = (a_1 \circ R_{0\to T})^\top x_0 + \sum_{t=1}^{T} (a_1 \circ R_{t\to T})^\top u_t \tag{55}$$

Substituting, we obtain the following:

$$
\begin{aligned}
F(w_T, x_T) &= F_0 + \sum_{t=1}^{T} \left\{ G_t^\top u_t - \frac{1}{2} u_t^\top P_t u_t \right\} - \frac{1}{2} x_T^\top a_2 \, x_T && (56) \\
F_0 &= w_0 R_{0,0\to T} + (a_1 \circ R_{0\to T})^\top x_0 && (57) \\
G_t &= a_1 \circ R_{t\to T} - \mathbf{1} \circ R_{0,t\to T} && (58) \\
P_t &= \Lambda_t \circ R_{0,t\to T} && (59)
\end{aligned}
$$

With these definitions, the objective function in equation (51) it can be rewritten as:

$$
F_0 - \frac{\gamma}{2} x_0^\top Q_0 x_0 + \max_{u_1,\ldots,u_T} \sum_{t=1}^{T} \mathsf{E}\left[G_t^\top u_t - \frac{1}{2} u_t^\top P_t u_t - \frac{\gamma}{2} x_t^\top Q_t x_t \right] \tag{60}
$$

subject to the non-linear dynamics given in equations (4) and (5) and where we have defined

$$
Q_t = \begin{cases} \Sigma_{t\to t+1} & \text{for } t < T \\ \frac{1}{\gamma} a_2 & \text{for } t = T \end{cases} \tag{61}
$$

Indeed, substituting the definition of our linear trading strategies from equations (24) and (25) into our objective function in equation (60) and then taking expectations gives:

$$
F_0 - \frac{\gamma}{2} x_0^\top Q_0 x_0 + \max_{\pi_1,\ldots,\pi_T} \sum_{t=1}^{T} \mathcal{G}_t^\top \pi_t - \frac{1}{2} \pi_t^\top \mathcal{P}_t \pi_t - \frac{\gamma}{2} \theta_t^\top \mathcal{Q}_t \theta_t \tag{62}
$$

$$
\text{subject to } \; \theta_t = \theta_{t-1}^0 + \pi_t \tag{63}
$$

and where we define the vector $\mathcal{G}_t$ and the square matrices $\mathcal{P}_t$ and $\mathcal{Q}_t$ for $t = 1, \ldots, T$ by

$$
\begin{aligned}
\mathcal{G}_t &= \mathsf{E}_0[\mathcal{B}_t G_t] && (64) \\
\mathcal{P}_t &= \mathsf{E}_0[\mathcal{B}_t P_t \mathcal{B}_t^\top] && (65) \\
\mathcal{Q}_t &= \mathsf{E}_0[\mathcal{B}_t Q_t \mathcal{B}_t^\top] && (66)
\end{aligned}
$$

Note that the time indices for $\mathcal{G}_t, \mathcal{P}_t, \mathcal{Q}_t$ also capture their size: $\mathcal{G}_t$ is a vector of length $NK(t+1)$, and $\mathcal{P}_t$ and $\mathcal{Q}_t$ are square matrices of the same dimensionality.[34] Equation (62) is just the objective function (equation (60)) with the u_t's and x_t's rewritten as linear functions

[34] It is important to note that these matrices $\mathcal{G}_t, \mathcal{P}_t, \mathcal{Q}_t$ will depend on the initial conditions (in particular on the initial exposures $\mathcal{B}_0$, which typically will depend on the initial positions in each stock).

of the elements in $\mathcal{B}_t$, with coefficients π_t and θ_t, respectively. Since the policy parameters π_t and θ_t are set at time 0, they can be pulled outside of the expectation operator.

Intuitively equation (62) is a linear-quadratic function of the policy parameters π_t and θ_t, with $\mathcal{G}_t$, $\mathcal{P}_t$, $\mathcal{Q}_t$ as the coefficients in this equation. These three components give, respectively, the effect on the objective function of: the expected portfolio returns resulting from trades at time t; the transaction costs paid as a result of trades at time t; and finally the effect of the holdings at time t on the risk-penalty component of the objective function.

Since $\mathcal{G}_t$, $\mathcal{P}_t$, $\mathcal{Q}_t$ are not functions of the policy parameters, they can be solved for explicitly or by simulation, and this only needs to be done once. Their values will depend on the initial conditions, and on the assumptions made about the state vector X_t driving the return generating process R_t and the corresponding security-specific exposure dynamics $B_{i,t}$. But, since equation (27) is a linear-quadratic equation, albeit a high-dimensional one, it can be solved using standard methods. We next calculate the closed form solution.

A.1 Closed form solution

We begin with the linear-quadratic problem defined by equations (62) and (63). Define recursively the value function starting from $V(T) = 0$ for all $t \leq T$ by:

$$V(t-1) = \max_{\pi_t} \left\{ \mathcal{G}_t^\top \pi_t - \frac{1}{2}\pi_t^\top \mathcal{P}_t \pi_t - \frac{\gamma}{2}\theta_t^\top \mathcal{Q}_t \theta_t + V(t) \right\}$$
$$\text{subject to} \quad \theta_t = \theta_{t-1}^0 + \pi_t$$

Then it is clear that $V(0)$ is the solution to the problem we are seeking. To solve the problem explicitly, we guess that the value function is of the form:

$$V(t) = -\frac{\gamma}{2}\theta_t^\top M_t \theta_t + L_t^\top \theta_t + H_t \tag{67}$$

with M_t a symmetric matrix. Since $V(T) = 0$, it follows that $M_T = 0$, $L_T = 0$ and $H_T = 0$. To find the recursion plug the guess in the Bellman equation:

$$V(t-1) = \max_{\pi_t} \left\{ \mathcal{G}_t^\top \pi_t - \frac{1}{2}\pi_t^\top \mathcal{P}_t \pi_t - \frac{\gamma}{2}\theta_t^\top (\mathcal{Q}_t + M_t)\theta_t + L_t^\top \theta_t + H_t \right\}$$
$$\text{subject to} \quad \theta_t = \theta_{t-1}^0 + \pi_t$$

Now plugging in the constraint, we can simplify the Bellman equation using the following notation $\overline{x}$ is the vector (submatrix) obtained from x by deleting the last NK rows (rows and columns). In Matlab notation $\overline{x} = x[1 : end - NK, 1 : end - NK]$.

$$V(t-1) = \max_{\pi_t} \Big\{ (\mathcal{G}_t + L_t)^\top \pi_t - \frac{1}{2}\pi_t^\top [\mathcal{P}_t + \gamma(\mathcal{Q}_t + M_t)]\pi_t - \frac{\gamma}{2}\theta_{t-1}^\top(\overline{\mathcal{Q}}_t + \overline{M}_t)\theta_{t-1}$$
$$-\gamma\theta_{t-1}^{0\top}[\mathcal{Q}_t + M_t]\pi_t + \overline{L}_t^\top\theta_{t-1} + H_t \Big\} \tag{68}$$

The first order condition gives:

$$\pi_t = [\mathcal{P}_t + \gamma(\mathcal{Q}_t + M_t)]^{-1}\left(\mathcal{G}_t + L_t - \gamma(\mathcal{Q}_t + M_t)^\top \theta_{t-1}^0\right),$$

and plugging into the state equation (equation (63)) we find

$$\theta_t = [\mathcal{P}_t + \gamma(\mathcal{Q}_t + M_t)]^{-1}\left(\mathcal{G}_t + L_t + \mathcal{P}_t^\top \theta_{t-1}^0\right).$$

Next, substitute these optimal policies into the Bellman equation in (68), giving:

$$V(t-1) = \frac{1}{2}(\mathcal{G}_t + L_t - \gamma(\mathcal{Q}_t + M_t)^\top\theta_{t-1}^0)^\top[\mathcal{P}_t + \gamma(\mathcal{Q}_t + M_t)]^{-1}\left(\mathcal{G}_t + L_t - \gamma(\mathcal{Q}_t + M_t)^\top\theta_{t-1}^0\right)$$
$$- \frac{\gamma}{2}\theta_{t-1}^\top(\overline{\mathcal{Q}}_t + \overline{M}_t)\theta_{t-1} + \overline{L}_t^\top\theta_{t-1} + H_t$$

Setting $\Psi_t = [\mathcal{P}_t + \gamma(\mathcal{Q}_t + M_t)]^{-1}$ and expanding we find:

$$\begin{aligned}
V(t-1) = H_t &+ \frac{1}{2}(\mathcal{G}_t + L_t)^\top\Psi_t(\mathcal{G}_t + L_t) \\
&- \gamma(\mathcal{G}_t + L_t)^\top\Psi_t(\mathcal{Q}_t + M_t)^\top\theta_{t-1}^0 + \overline{L}_t^\top\theta_{t-1} \\
&- \frac{\gamma}{2}\theta_{t-1}^\top\left[\overline{\mathcal{Q}}_t + \overline{M}_t - \gamma(\overline{\mathcal{Q}}_t + \overline{M}_t)^\top\overline{\Psi}_t(\overline{\mathcal{Q}}_t + \overline{M}_t)\right]\theta_{t-1}
\end{aligned}$$

Comparing this equation and the conjectured specification for $V(t)$ in equation (67) shows that this specification will be correct if H_t, L_t, and M_t are chosen to satisfy the recursions:

$$\begin{aligned}
H_{t-1} &= H_t + \frac{1}{2}(\mathcal{G}_t + L_t)^\top\Psi_t(\mathcal{G}_t + L_t) \\
L_{t-1} &= \overline{L}_t - \gamma\overline{(\mathcal{Q}_t + M_t)\Psi_t(\mathcal{G}_t + L_t)} \\
M_{t-1} &= \overline{\mathcal{Q}}_t + \overline{M}_t - \gamma(\overline{\mathcal{Q}}_t + \overline{M}_t)^\top\overline{\Psi}_t(\overline{\mathcal{Q}}_t + \overline{M}_t)
\end{aligned}$$

with initial conditions $H_T = 0$, $L_T = 0$ and $M_T = 0$.

We have thus derived the optimal value function and the optimal trading strategy in the LGS class.

Before discussing some specific examples it is useful to introduce a set of LGS strategies which uses the exposures lagged at most ℓ periods. This set of 'restricted lag' LGS is useful in applications when the time horizon is fairly long, and for signals that have a relatively fast decay rate, so that the dependence on lagged exposures can be restricted without a significant cost. We next show that the same tractability obtains for the restricted lag setting.

B Constant variance of returns versus price changes

B.1 In dollars

Suppose x_t is vector of dollar holdings in risky shares and u_t is dollar trade at time t. R_f is the risk-free rate and R_t is the vector of Gross returns. The net returns are given by $r_t = R_t - \mathbf{1}$ and $r_f = R_f - 1$.

Then we have with the convention that we trade at the end of the period:

$$
\begin{aligned}
x_{t+1} &= x_t . * R_{t+1} + u_{t+1} && (69)\\
W_{t+1} &= W_t R_f + x_t'(R_{t+1} - R_f) - \frac{1}{2} u_{t+1} \Lambda_d u_{t+1} && (70)
\end{aligned}
$$

B.2 In shares

Suppose n_t is vector of number of shares held in risky shares and h_t is number of shares traded at time t. R_f is the risk-free rate and $dS_{t+1} = S_{t+1} - S_t$ is the vector of price changes (Assume no dividends for simplicity).

Then we have with the convention that we trade at the end of the period:

$$
\begin{aligned}
n_{t+1} &= n_t + h_{t+1} && (71)\\
W_{t+1} &= W_t R_f + n_t'(dS_{t+1} - r_f S_t) - \frac{1}{2} h_{t+1} \Lambda_s h_{t+1} && (72)
\end{aligned}
$$

B.3 The objective function

For simplicity we set $r_f = 0$ and as in GP we solve the infinite horizon problem where the investor maximizes the discounted value of mean-variance objective functions.

In dollars

$$
\mathsf{E}\left[\sum_{t=1}^{\infty} \rho^t \left\{ x_t \mu_d - \frac{1}{2} u_t \Lambda_d u_t - \frac{\gamma}{2} x_t' \Sigma_d x_t \right\}\right] \quad (73)
$$

or, equivalently, in shares:

$$\mathsf{E}\left[\sum_{t=1}^{\infty}\rho^t\left\{n_t\mu_s - \frac{1}{2}h_t\Lambda_s h_t - \frac{\gamma}{2}n_t'\Sigma_s n_t\right\}\right]$$

Now, note that by definition:

$$x_t = n_t.*S_t \quad (74)$$

$$u_t = h_t.*S_t \quad (75)$$

$$\mu_s = \mu_d.*S_t \quad (76)$$

$$\Sigma_s = I_{S_t}\Sigma_d I_{S_t} \quad (77)$$

$$\Lambda_s = I_{S_t}\Lambda_d I_{S_t} \quad (78)$$

So clearly, assuming that the expectation and variance of dollar returns are constant is inconsistent with assuming that the expectation and variance of price changes are constant. We compare both cases next.

B.4 Constant expectation and variance of dollar returns

Let's assume that the expectation and variance of returns are constant. Then it is helpful to introduce the state variable $\overline{x}_t = x_t - u_t$, so that

$$\overline{x}_{t+1} = (\overline{x}_t + u_t).*R_{t+1} \quad (79)$$

We can define the value function recursively by:

$$J(\overline{x}_t) = \max_{u_t}\left\{(\overline{x}_t + u_t)\mu_d - \frac{1}{2}u_t\Lambda_d u_t - \frac{\gamma}{2}(\overline{x}_t + u_t)'\Sigma_d(\overline{x}_t + u_t) + \rho\mathsf{E}_t[J(\overline{x}_{t+1})]\right\} \quad (80)$$

Guess that the value function is quadratic.

$$J(\overline{x}) = M_0 + M_1'\overline{x} + \overline{x}'M_2\overline{x}$$

Let's first consider the one risky asset case. Then the solution is simply:

$$u_t + \overline{x}_t = \frac{\overline{x}_t\Lambda_d + \mu_d + M_1\rho\mu_d}{\Lambda_d + \gamma\Sigma_d - 2M_2\rho(\mu_d^2 + \Sigma_d)} =: a_0 + a_1\overline{x}_t \quad (81)$$

where the coefficient of the optimal value function are given by:

$$M_2 = -\frac{\sqrt{\left(\gamma\Sigma - \Lambda\left(\rho\left(\mu^2+\Sigma\right)-1\right)\right)^2 + 4\gamma\Lambda\rho\Sigma\left(\mu^2+\Sigma\right)} - \gamma\Sigma + \Lambda\left(\rho\left(\mu^2+\Sigma\right)-1\right)}{4\rho\left(\mu^2+\Sigma\right)} \quad (82)$$

$$M_1 = \frac{2\Lambda\mu}{\sqrt{\left(\gamma\Sigma - \Lambda\left(\rho\left(\mu^2+\Sigma\right)-1\right)\right)^2 + 4\gamma\Lambda\rho\Sigma\left(\mu^2+\Sigma\right)} + \gamma\Sigma + \Lambda\mu^2\rho - 2\Lambda\mu\rho + \Lambda\rho\Sigma + \Lambda} \quad (83)$$

and M_0 can be computed explicitly, but is a rather lengthy expression.[35] Note that

$$a_1 = \frac{2\Lambda_d}{\Lambda(1+\rho\left(\mu^2+\Sigma\right)) + \gamma\Sigma_d + \sqrt{\left(\gamma\Sigma - \Lambda\left(\rho\left(\mu^2+\Sigma\right)-1\right)\right)^2 + 4\gamma\Lambda\rho\Sigma\left(\mu^2+\Sigma\right)}}$$

Simple algebra confirms that $a_1 \in (0,1)$ if $\gamma\Lambda\Sigma\rho > 0$.

B.5 Constant expectation and variance of price changes

For comparison purposes we make the same change of variables $\overline{n}_t = n_t - h_t$ so that

$$\overline{n}_{t+1} = \overline{n}_t + h_t$$

Then we define the value function recursively by:

$$J(\overline{n}_t) = \max_{h_t}\left\{(\overline{n}_t + h_t)\mu_s - \frac{1}{2}h_t\Lambda_s h_t - \frac{\gamma}{2}(\overline{n}_t + h_t)'\Sigma_s(\overline{n}_t + h_t) + \rho\mathsf{E}_t[J(\overline{n}_{t+1})]\right\} \quad (84)$$

Guess that the value function is quadratic.

$$J(x) = N_0 + N_1'\overline{n} + \overline{n}'N_2\overline{n}$$

Let's first consider the one risky asset case. Then we can solve everything in closed-form and we obtain:

$$h_t + \overline{n}_t = \frac{\overline{n}_t\Lambda_s + \mu_s + N_1\rho}{\Lambda_s + \gamma\Sigma_s - 2N_2\rho} \quad (85)$$

where the coefficient of the optimal value function are given by:

[35] All calculations were made in Mathematica and the file is available upon request.

$$N_2 = \frac{-\sqrt{(\gamma\Sigma + \Lambda(-\rho) + \Lambda)^2 + 4\gamma\Lambda\rho\Sigma} + \gamma\Sigma + \Lambda(-\rho) + \Lambda}{4\rho} \tag{86}$$

$$N_1 = \frac{2\Lambda\mu}{\sqrt{(\gamma\Sigma + \Lambda(-\rho) + \Lambda)^2 + 4\gamma\Lambda\rho\Sigma} + \gamma\Sigma + \Lambda(-\rho) + \Lambda} \tag{87}$$

and

$$N_0 = \left\{ -\frac{\mu^2 \left((\rho-1)\sqrt{\gamma^2\Sigma^2 + 2\gamma\Lambda(\rho+1)\Sigma + \Lambda^2(\rho-1)^2} + \gamma(\rho+1)\Sigma + \Lambda(\rho-1)^2 \right)}{4\gamma^2(\rho-1)\rho\Sigma^2} \right\} \tag{88}$$

B.6 Comparing the two solutions

The most obvious difference between the two solutions is that in the "constant expectation and variance of price change" case there exists a no-trade solution.

Indeed, solving for the fixed point $\overline{n}_t$:

$$\frac{\overline{n}_t\Lambda_s + \mu_s + N_1\rho}{\Lambda_s + \gamma\Sigma_s - 2N_2\rho} = \overline{n}_t$$

which is equivalent to

$$n_{no} = \frac{\mu}{\gamma\Sigma} \tag{89}$$

then we see that if $\overline{n}_t = n_{no}$ at some time t, then it is optimal to **NEVER** trade from then on, since $h_t = 0$ and therefore $\overline{n}_{t+s} = \overline{n}_{t+1} = \overline{n}_t = n_{no}$ $\forall s > 0$ by induction. Instead, in the "constant expectation and variance of return" case, we see that the system can never settle into a no-trade equilibrium, since the dynamics of the state always lead to $\overline{x}_{t+1} \neq \overline{x}_t$ even if $u_t = 0$.

Further, it is interesting to note that the state where it is optimal not to trade **for one period** at time t in the "constant expectation and variance of return" case, is actually NOT the mean-variance efficient portfolio. Indeed, the no trade position for that case corresponds to a dollar position such that:

$$\overline{x}_t = \frac{\overline{x}_t\Lambda_d + \mu_d + +M_1\rho\mu_d}{\Lambda_d + \gamma\Sigma_d - 2M_2\rho(\mu_d^2 + \Sigma_d)}$$

Solving for x_{no} we find:

$$x_{no} = \frac{2\mu\left(\mu^2+\Sigma\right)}{((\mu-1)\mu+\Sigma)\sqrt{\gamma^2\Sigma^2+2\gamma\Lambda\Sigma\left(\rho\left(\mu^2+\Sigma\right)+1\right)+\Lambda^2\left(\rho\left(\mu^2+\Sigma\right)-1\right)^2}+\gamma\Sigma\left(\mu^2+\mu+\Sigma\right)+\Lambda((\mu-1)\mu+\Sigma)\left(\rho\left(\mu^2+\Sigma\right)-1\right)} \tag{90}$$

Note that $x_{no} = \frac{\mu_d}{\gamma\Sigma_d}$ if $\Lambda_d = 0$ or if $\rho = 0$, but otherwise it is different!

Further, even if $x_t = x_{no}$ at some t and thus $u_t = 0$ is optimal, since $\overline{x}_{t+1} = \overline{x}_t R_{t+1}$ in that case, it will become optimal to trade at time $t+1$.

C Calibration of the Simulation Experiment

The RGPs for the characteristics and the factor environments (equations (40) and (41)) are, respectively

$$R_{i,t+1} = \beta_{i,t}^\top (F_{t+1} + \lambda) + \epsilon_{i,t+1}$$

where $\mathsf{E}_t[F_{t+1}] = 0$ and $\mathsf{E}_t[F_{t+1}F_{t+1}^\top] = \Omega$ and

$$R_{i,t+1} = \beta_{i,t}^\top \lambda + \nu\epsilon_{i,t+1},$$

where the factor exposures $\beta_{i,t}$ and premia λ are each $(K, 1)$ vectors, and and where the evolution of the factor exposures is given by equation (40):

$$\beta_{i,t+1}^k = (1-\phi_k)\beta_{i,t}^k + \epsilon_{i,t+1},$$

or equivalently:

$$\beta_{i,t}^k = \sum_{s=0}^{\infty}(1-\phi_k)^s \epsilon_{i,t-s}.$$

Taken together, these imply, for either environment, that:

$$\begin{aligned} \mathsf{E}_t\left[R_{i,t+1}\right] &= \beta_{i,t}^\top \lambda \\ &= \sum_{k=1}^{K} \lambda_k \beta_{i,t}^k \\ &= \sum_{k=1}^{K} \lambda_k \sum_{s=0}^{\infty} (1-\phi_k)^s \epsilon_{i,t-s}. \end{aligned}$$

In our simulation experiment in Section 3, we model the return-generating process for equities as consisting of $K = 3$ factors consistent with the short-term-reversal, mediumterm-momentum, and long-term-reversal effects. Consistent with the evidence on these three effect, we choose half-lives for these factors of 5 days, 150 days, and 700 days.

To determine the parameters λ and Ω, we calibrate this factor model using the monthly returns of portfolios formed on the basis of momentum, short- and long-term reversal, available on Ken French's website. We use the full sample, 1927:01-2013:12. Note that data is available on both the pre-formation and the post-formation returns of these sets of portfolios. We perform a Fama-MacBeth-like regression of the post-formation returns on the pre-formation returns for each of the three sets of decile portfolios, and use the resulting coefficients to estimate the set of λs, given our assumed set of ϕs.

We characterize the slope coefficients for the three regressions with the formation period return horizons: our notation is that the formation period, for regression $j \in \{str, mom, ltr\}$, runs from time $t-m_j$ to $t-n_j$. For the characteristics model, the (cross-sectional) projection of a one-day return onto a sum of returns from time $t-m_j$ to $t-n_j$ will give, under the assumptions of our model.[36]

$$\begin{aligned} cov\left(R_{i,t+1}, \sum_{s=n_j}^{m_j} \epsilon_{i,t-s}\right) &= \sigma_\epsilon^2 \sum_{k=1}^{3} \lambda_k \beta_{i,t}^k \\ &= \sigma_\epsilon^2 \sum_{k=1}^{3} \sum_{s=n_j}^{m_j} \lambda_k (1-\phi_k)^s \end{aligned}$$

and

$$var\left(\sum_{s=n_j}^{m_j} \epsilon_{i,t-s}\right) = (m_j - n_j + 1)\sigma_\epsilon^2.$$

and finally

$$\begin{aligned} \beta_j &= \frac{cov\left(R_{i,t+1}, \sum_{s=n_j}^{m_j} \epsilon_{i,t-s}\right)}{var\left(\sum_{s=n_j}^{m_j} \epsilon_{i,t-s}\right)} = \sum_{k=1}^{3} \lambda_k \frac{1}{(m_j - n_j + 1)} \sum_{s=n_j}^{m_j} (1-\phi_k)^s. \\ &= \sum_{k=1}^{3} \left(\frac{(1-\phi_k)^{n_j} - (1-\phi_k)^{m_j+1}}{\phi_k (m_j - n_j + 1)} \right) \lambda_k \\ &= \sum_{k=1}^{3} a_{j,k} \lambda_k \end{aligned}$$

[36] In practice we actually calculate the betas using returns rather than residuals. However, given that, in the data particularly at short horizons, most of the variance of returns is idiosyncratic as opposed to expected return variation, this approximation seems reasonable.

where

$$a_{j,k} = \left(\frac{(1-\phi_k)^{n_j} - (1-\phi_k)^{m_j+1}}{\phi_k(m_j - n_j + 1)} \right) \tag{91}$$

We find the three values of λ_k by solving the set of linear equations (for the three empirically estimated β_js).

$$\begin{bmatrix} \beta_{str} \\ \beta_{mom} \\ \beta_{ltr} \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

λ Estimation:

The Fama-MacBeth regressions yield (average) coefficients of:

$$\begin{bmatrix} \beta_{str} \\ \beta_{mom} \\ \beta_{ltr} \end{bmatrix} = \begin{bmatrix} -0.00116273 \\ 0.00044366 \\ -0.00010126 \end{bmatrix}$$

The resulting λ estimates are:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} -0.093482 \\ 0.001484 \\ -0.000400 \end{bmatrix}$$

Ω Calibration:

The goal in the Ω calibration is to come up with an upper bound on the magnitude of the covariance matrix. We employ the following procedure to estimate the 3×3 factor covariance matrix Ω using the three sets of decile portfolio returns: *str*, *mom*, and *ltr*.

First, we use only the excess returns of the zero-investment portfolios formed by going long the top decile and short the bottom decile (*i.e.*, the 10–1 portfolios). The factor loadings for these excess return portfolios are (from equation (40)

$$\beta^{10-1}_{j,k,t} = \sum_{s=0}^{\infty} (1-\phi_k)^s \epsilon^{10-1}_{j,t-s}$$

Here, $j \in \{str, mom, ltr\}$ is French's portfolio formation method; $k \in \{1, 2, 3\}$ is the factor identifier, and t is the time (end-of-period) at which we are measuring the factor loading. As in the preceding section, $t - n_j$ and $t - m_j$ are the starting and ending times for the period over which the pre-formation returns are measured for portfolio j.

We are going to make several assumptions to allow the calculation of the factor loadings for each of these three portfolios. First, since portfolio j is formed on the basis of individual firm returns from $t - m_j$ to $t - n_j$, we assume that the residual returns for the portfolios are zero outside of that time range. This means that:

$$\beta_{j,k,t}^{10\text{–}1} = \sum_{s=n_j}^{m_j} (1 - \phi_k)^s \epsilon_{j,t-s}^{10\text{–}1}$$

Second, note that French only provides the formation period return on an annual basis. So, for example, for the LHR portfolios we have their cumulative return from t-60 months through t-12 months. So we assume that the average return was earned equally over each day in the 48 month period. If we denote the total pre-formation return as R^{pre}, we assume that the daily return, for each day in the 4 year period, was $R^{pre}/(4 \cdot 252)$. In general, given a $10-1$ differential pre-formation return for strategy j in year y of $R_{j,y}^{pre,10\text{–}1}$, we calculated the each daily return over the formation period as:

$$R_{j,s}^{pre,10\text{–}1} = \frac{R_{j,y}^{pre,10\text{–}1}}{(m_j - n_j + 1)}$$

for each day s between $t - m_j$ and $t - n_j$, and zero outside of the formation period. This means that the factor loading for portfolio $10-1$ portfolio j on factor k is:

$$\begin{aligned}
\beta_{j,k,t}^{10\text{–}1} &= \frac{R_{j,y}^{pre,10\text{–}1}}{(m_j - n_j + 1)} \sum_{s=n_j}^{m_j} (1 - \phi_k)^s \quad \forall t \in y \\
&= \left(\frac{(1 - \phi_k)^{n_j} - (1 - \phi_k)^{m_j+1}}{\phi_k (m_j - n_j + 1)} \right) R_{j,y}^{pre,10\text{–}1} \quad \forall t \in y \\
&= a_{j,k} R_{j,y}^{pre,10\text{–}1}
\end{aligned}$$

where $a_{j,k}$ is defined in equation (91).

Next, we assume that, since these are relatively well diversified portfolios, the residual risk (σ_ϵ^2) is zero and further assume that all of the systematic risk comes from the three factors. These two assumptions imply that the covariance matrix for the time $t + 1$ returns of the three $10-1$ portfolios, which we denote $\mathbf{\Sigma}_t$, is given by:

$$\mathbf{\Sigma}_t = \boldsymbol{\beta}_t \Omega_t \boldsymbol{\beta}_t^\top$$

where

$$\boldsymbol{\beta}_t = \begin{bmatrix} \beta^{10\text{–}1}_{str,1,t} & \beta^{10\text{–}1}_{str,2,t} & \beta^{10\text{–}1}_{str,3,t} \\ \beta^{10\text{–}1}_{mom,1,t} & \beta^{10\text{–}1}_{mom,2,t} & \beta^{10\text{–}1}_{mom,3,t} \\ \beta^{10\text{–}1}_{ltr,1,t} & \beta^{10\text{–}1}_{ltr,2,t} & \beta^{10\text{–}1}_{ltr,3,t} \end{bmatrix}$$

Note that this system is just identified, and Ω is given by:

$$\Omega = \left(\boldsymbol{\beta}_t^\top \boldsymbol{\beta}_t\right)^{-1} \boldsymbol{\beta}_t^\top \mathbf{\Sigma}_t \boldsymbol{\beta}_t \left(\boldsymbol{\beta}_t^\top \boldsymbol{\beta}_t\right)^{-1}$$

We can estimate this either using the full sample covariance and the average pre-formation returns, or year-by-year and average the results.

Over the full-sample the average daily volatility of the daily $10-1$ portfolio returns are (in basis points):

$$\begin{bmatrix} \sigma_{str} \\ \sigma_{mom} \\ \sigma_{lhr} \end{bmatrix} = \begin{bmatrix} 28.464 \\ 37.817 \\ 30.367 \end{bmatrix}$$

and the correlation matrix of the returns is:

$$\begin{bmatrix} 1 & 0.250744 & 0.087098 \\ 0.250744 & 1 & 0.333539 \\ 0.087098 & 0.333539 & 1 \end{bmatrix}$$

The factor loading matrix for these three portfolios is:

$$\mathbf{B} = \begin{bmatrix} 0.007291874 & 0.2927041 & 0.3146322 \\ 1.974574 \times 10^{-05} & 0.6481128 & 1.0529 \\ 1.061207 \times 10^{-28} & 0.2732635 & 2.100848 \end{bmatrix} \tag{92}$$

giving an estimated $\hat{\Omega}$ of:

$$\hat{\Omega} = \begin{bmatrix} 0.1655572 & -0.001041718 & 0.000119914 \\ -0.001041718 & 4.898553 \times 10^{-05} & -7.10805 \times 10^{-06} \\ 0.000119914 & -7.10805 \times 10^{-06} & 3.109768 \times 10^{-06} \end{bmatrix}$$

Or, decomposing this, the (daily) factor volatilities are:[37]

$$\hat{\sigma}_f = \begin{bmatrix} 0.4068872 \\ 0.0069990 \\ 0.0017635 \end{bmatrix}$$

and the correlation matrix of the factors is estimated to be:

$$\hat{\rho} = \begin{bmatrix} 1 & -0.3657987 & 0.1671214 \\ -0.3657987 & 1 & -0.5759073 \\ 0.1671214 & -0.5759073 & 1 \end{bmatrix}$$

[37] Note that the first factor has a large volatility (40%/day). This is a result of the way that we define the factor loadings in equation (40), where a firm's factor loading is an exponentially weighted sum of past residual returns. When ϕ^k is large, as it is for $k = 1$, the dispersion in factor loadings across firms in the economy will be small. This is apparent in equation (92). Thus, a large factor volatility is required to explain the the volatility of the long-short str volatility of only 28 bp/days.