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A Theory of Costly Sequential Bidding*

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Abstract. We model sequential bidding in a private value English auction when it is costly to submit or revise a bid. We show that, even when bid costs approach zero, bidding occurs in repeated jumps, consistent with certain types of natural auctions such as takeover contests. In contrast with most past models of bids as valuation signals, every bidder has the opportunity to signal and increase the bid by a jump. Jumps communicate bidders' information rapidly, leading to contests that are completed in a few bids. The model additionally predicts informative delays in the start of bidding, that the probability of a second bid decreases in, and the jump increases in the first bid, that objects are sold to the highest valuation bidder; and revenue and efficiency relationships between different auctions.

1. Introduction

Several markets for unique and valuable objects proceed as variations of English auctions, involving a sequence of ascending bids terminated when bidders are unwilling to submit further bids.[1] Existing theory of private value English auctions has identified a single equilibrium described by Vickrey (1961), in which each bidder, in turn, submits a bid equal to the previous bid plus the minimum bid increment unless the resulting bid would be higher

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[1] Examples include both formally organized auctions such as Sotheby's, spontaneous auctions such as takeover contests, and (with modification) some privatizations such as the U.S. government auctions for rights to personal communication services spectrum. McMillan (1994) provides a good summary of the PCS spectrum auction design.

than his valuation, in which case he passes. We refer to this outcome as the *ratchet solution*.

We show that the ratchet solution is just one among a continuum of equilibria of the costless English auction. In this paper we view sequential bidding as a *learning process*. Different equilibria have different rates of revelation about bidders' valuations, and therefore different rates of completion of the auction. The ratchet equilibrium has a rather extreme property: it *minimizes* the rate at which bidders learn about each others' valuations (subject to the constraint that some revelation occurs at every bid). Under the ratchet solution, each bid exceeds the preceding one by a minimal increment, and thereby rules out a minimal interval of possible valuations between the bidder's previous and most recent bid. In contrast, this paper focuses on a signalling equilibrium that *maximizes* the rate of relevant learning.

In this signalling equilibrium, bidding occurs in a series of highly informative jumps, and terminates after only a few bids. At each point along the equilibrium path, a bidder either quits or jumps to a bid level so high that his opponent is deterred from continuing further. Each bid is either fully revealing, or else reveals information so favorable about the bidder that his opponent is deterred with certainty. In this sense the rate of relevant learning is maximized.

Something like the ratchet solution is the only sensible outcome when the cost of bidding is zero. The premise of the ratchet solution is that a bidder should always continue to bid until his valuation is reached. Even if he knows his valuation is lower than a competitor, the bidder may as well make another try, since he has nothing to lose by doing so and conceivably may gain. Thus, any strategy that involves bid jumps for the purpose of signalling high valuation creates a risk of unnecessarily bidding higher than the competitor's valuation, and cannot intimidate a competitor into quitting early.

Based on casual observation, this prediction is broadly consistent with observed behavior in English auction settings where costs are negligible. In such settings, bidders all sit in the same room and increase bids sequentially. This is a setting with especially low costs of raising the bid, as attendees are already devoting their time and attention to the auction, and all that is needed to increase the bid is a hand gesture. With bid costs very close to zero, our theory suggests that the Ratchet Solution may be feasible. Our model therefore helps explain why we can sometimes observe a gradually ratcheting bidding process (at auction houses), but seldom observe this in settings such as takeover contests where bid costs are nonnegligible.

We argue that the reason for the distinctly different behavior in these two kinds of sequential auctions is the difference in the costs associated with bidding. In our model, we specify that there is purely dissipative cost that must be paid when a bid is submitted (or revised). Paying this cost does not provide the bidder with benefits or any information, but is required to bid. We show that, in this setting, an equilibrium exists in which each bidder prefers to jump over the previous bid, so as to *signal* high valuation.[2] To see why, consider the situation where a bidder is convinced that his opponent is going to win (because of high valuation). In this case he strictly prefers to quit immediately rather than incur the cost of another bid. For example, if bidders know each others valuations, the lower valuation bidder will quit immediately rather than waste the bid cost (see Hirshleifer and Png (1990)). In this equilibrium, the auction allocates the object to the highest value bidder in only a few bids, and thereby economizes on bid costs.

In the model we provide here, our underlying assumption is that submitting a bid is costly, but that "passing" (waiting without bidding) is costless. Clearly this is a stylized assumption. While the M&A transaction process is distracting for a bidder, waiting without bidding can also have an attention cost. However, we view both the direct costs and the indirect costs (e.g, the attention costs) of having the transaction be "live" is higher, owing to greater legal costs, and to a need to deal with questions from the board, shareholders, and the media, and of interacting with advisors. Although such costs are potentially present for an offer or revision that is only being contemplated, these costs are likely to be much higher for actually making a bid. As long as there are incremental costs associated with submitting a bid, a low-valuation potential acquirer can benefit from passing if he is sure to lose, or, in less extreme cases, waiting to see if other potential acquirers are willing to bid. Thus, while in our model the cost of passing is zero, similar results are likely to apply even in settings with positive costs to passing.

There are two benefits of signalling (bidding high) related to forcing out competitors. First, deterring a competitor whose valuation is above the bid of the signalling bidder allows that bidder to buy at a lower price. Second, driving out a competitor early reduces the expected bid costs to be incurred. The cost of signalling is that a bidder may pay more than was necessary to drive out a competitor whose valuation is below the signalling bid.

The nature of the equilibrium that we examine is as follows. Based on his valuation, the first bidder assesses his chance of winning the auction. Provided this probability is sufficiently high, he makes a bid; otherwise he passes. If he bids, the level of the bid reveals his valuation. The other bidder then either passes, ending the auction, or jumps to a higher bid that signals

[2] Several papers have recognized the empirical importance of jumps. Most of these papers modify the ratchet solution to allow for a single jump bid. We discuss the relation of our paper to other work in Section 7.

that his valuation is at least as high as the initial bidder. If so, then the first bidder passes, ending the auction.

If the first bidder passes, then the second bidder decides whether his valuation is sufficiently high that it pays for him to initiate the bidding. In so doing he takes into account the fact that the first bidder's valuation is below the equilibrium critical value needed to initiate bidding. If the second bidder's valuation is above his own critical value, he initiates the bidding, and otherwise he passes. If he does bid, he will again make a jump bid that provides a signal about his valuation. Just as in the case where the first bidder initiates the bidding, the auction ends after either one or two bids.[3] In a similar fashion, if the second bidder's valuation is below his critical value the first bidder will have another opportunity to open the bidding, and so on.

Thus, in this equilibrium bidders with low valuations sometimes delay their bids in order to assess the strength of their competition. So long as there is at least one bidder whose valuation exceeds the bid cost, eventually a bid occurs, but there may be any number of rounds of delay before the bidder becomes confident enough of his chances of winning to make an opening offer.

This signalling equilibrium is consistent with the empirical reality that from first bid to last, jumps are common; this behavior is not implied by the ratchet solution, or by a model with an up-front entry/investigation cost (see *e.g.*, Fishman (1988), discussed below). Such models predict that, apart from a possible initial jump, bids will increase by numerous minimally informative increments (e.g., a one cent increase on each bid) until one bidder quits. In practice, bidding for corporate acquisitions typically moves in large jumps, and ends after a few bids (Betton and Eckbo, 2000; Dimopoulos and Sacchetto, 2014).[4] Cramton (1997) provides evidence of frequent large jumps

[3] If the second bidder's valuation is not too high, his bid may perfectly reveal his valuation. However, if his valuation is sufficiently high, he will bid just high enough to drive out the first bidder with certainty. This is a possibility since the first bidder has revealed low valuation by his failure to bid at the first opportunity.

[4] The *Wall Street Journal* reports large initial bid premia in several takeovers. For example, Mattel offered a 73 percent premium ($2.2 billion) in a $5 billion unsolicited bid for Hasbro (*WSJ* 1/25/96), and Johnson & Johnson's August 1994 agreement to buy Neutrogena at $35.25 per share was a 70 percent premium over the price two weeks before the bid. In an article entitled "Whopping Initial Bids Become Trend of 90's," Sandoz's 1994 bid for Gerber was for $53, compared to a preceding day price of $35; several similar transactions were reported. Jumps in bidding are common after the initial bid as well. For example, the takeover bidding for Conoco in 1981 saw an initial bid of $70 per share by Seagram's over a market stock price of $58.875. This was followed several weeks later by a competing bid of $87.50 from a Conoco management/DuPont group, and 11 days later by a $90 per

in the bidding for personal communication spectrum (PCS) rights auctioned by the U.S. government. Observers of other auctions have commented on similar phenomenon, such as Cassady (1967, p. 75), who observes about private auctions that "... [the would-be buyer] may offer a high price at an early stage in the proceedings in the hope of scaring off competitors." Easley and Tenorio (2004) document extensive jump bidding in internet auctions, and that this strategy is effective in deterring competitors.

Furthermore, the conventional ratchet analysis does not examine incentives to wait to see what competitors do before making an offer. In practice, delays in opening the bidding are not uncommon. In formal auctions, if there are no bids at the required opening bid, the auctioneer will often lower the required opening level until he hears a bid. After this, the bid sometimes then progresses to a level higher than the initial required opening bid (Cassady, 1967). Similarly, in the market for corporations, takeover rumors about possible bidders for a target firm often circulate for significant periods of time before an offer appears, followed in some cases by competition between multiple bidders for the target. For example, Paramount was the subject of takeover rumors for several months before Viacom first announced its bid in September 1993; one week later, QVC announced a higher bid.

Our approach suggests that a reason for these discrepancies between theoretical predictions and empirical observations may be that, in some spontaneous auctions, submitting or revising a bid is costly. Hirshleifer and Png (1990) suggest that the costs of takeover bidding include "... fees to counsel, investment bankers, and other outside advisors, the opportunity cost of executive time, [and] the cost of obtaining financing for the bid." Also, in the U.S. some mandated S.E.C. information filings have to be repeated with each bid revision.[5] Seyhun (1997, p. 296) reports that unsuccessful takeover bidders experience stock returns of -0.7 percent, in contrast with positive 0.7 percent for successful bidders, a significant difference. He argues that "If the bidder firms do not succeed, they are stuck with the costs while they do not enjoy the synergistic benefits of the takeover."

We apply the model to derive a number of implications about the determinants of delay; the information conveyed by such delay; the relations between

share bid from Mobil (see Ruback (1982)). Bidding in the 1982 takeover contest for Cities Service went from a stock price of $35.50 to an initial bid of $45, to a competing bid of $63 (see Ruback (1983)). And in 1984-85, bidding for Unocal jumped from a stock price of $48 to an initial offer of $54, to a competing bid of $78 Weston, Chung, and Hoag (1990, p. 616-19, 522).

[5] In addition, if the takeover will be associated with restructuring of the bidder and target, then real investment and operating decisions of the bidder may be hampered by continuing uncertainty over whether merger will occur.

bidding schedules, bidder profits, and seller revenues in the signalling equilibrium of the costly sequential bidding (CSB) auction with other more familiar auctions (with entry fees or minimum bids); the asymptotic optimality of the CSB auction when bid costs are small; bidder preferences regarding order of moves; and the predicted empirical relationship between initial bid jumps, competition and subsequent jumps.

First, the model predicts that in costly sequential auctions, bidding will proceed with substantial jumps at each stage. Second, the probability of a second bidder making an offer is decreasing in the level of the first bid. Third, under mild assumptions, the jump between the first and the second bid is increasing in the initial bid. Fourth, bidders will sometimes wait for long periods of time before entering an auction, with adverse information about valuations publicly revealed by the amount of delay. This could be tested by examining whether bids that occur after a longer delay (for example, a longer time after takeover rumors begin) tend to be lower. This could also potentially be tested by examining whether the prices of a potential target decline after a takeover rumor as more time goes by without any bid. Fifth, objects are sold to the highest valuation bidder (which is, at a minimum, testable in experimental settings). Sixth, the length of delays are increasing with bid cost. Seventh, a first price sealed bid auction yields approximately the same revenue for the seller as a CSB auction, the difference being on the order of bid costs. This offers a possible explanation for why takeover auctions are often allowed to proceed with a spontaneous CSB auction rather than a designed optimal auction.

The sequential English auction is the spontaneous format that arises if no special effort is taken to design an auction mechanism: each bidder either raises the bid or loses. We focus on this specific mechanism because we are interested in modeling the consequences of an auction that is very frequently used.[6] There are several possible reasons why this mechanism is in fact used, such as problems of time-consistency, and the fact that publicizing a

[6] Furthermore, any other format is potentially subject to a time-consistency problem wherein the seller is unable to commit to a mechanism for an auction: consistent with our sequential structure, the seller will always consider a higher bid. For example, the noted takeover advisor Bruce Wasserman stated “Naturally, sophisticated bidders will do their best to circumvent the auction format. ... Sometimes a bidder will raise its bid after the final deadline despite the rules. The auctioneer is then in a quandary and sometimes invites another round of bids. Obviously, the original ‘winning’ bidder will be furious.” Wasserstein (1998, pp. 9-28) describes the bidding by QVC (Barry Diller) and Viacom (Sumner Redstone) for Paramount (Marvin Davis). After the final bids had been submitted for the ‘final’ deadline on 4pm on December 20, 1993, Viacom returned with a further bid in January. Later Viacom and QVC upped the bids still further, and Viacom eventually won.

mechanism may legally compel management to sell the firm when it would prefer not to do so.[7] Our approach suggests a further reason. We show that the seller's expected revenue in the 'spontaneous' sequential auction will be arbitrarily close to what would be expected from the optimal mechanism when bid costs are small. Thus, if there are nontrivial costs involved in setting up an optimal mechanism, we often expect to see the spontaneous auction to be used.

The remainder of the paper is structured as follows. Section 2 outlines the economic setting, and Section 3 reviews the ratchet solution. Section 4 examines the equilibrium when bidding is costless, and Section 5 the equilibrium with costs. Section 6 compares the sequential auction with other auctions. Section 7 relates our results to others in the auction and takeover literature, and Section 8 concludes.

2. The Economic Setting

A single good is to be auctioned to two potential bidders. The i'th bidder's valuation for the good, θ_i, is independent of the other bidder's valuation, and its distribution is given by the strictly increasing and twice differentiable probability distribution function $F_i(\theta)$ on $[\underline{\theta}, \overline{\theta}]$, where $\underline{\theta} \geq 0$. (Our assumption that all individuals' distributions have the same upper and lower bounds is mainly to simplify the notation.) Bidders are assumed to maximize their expected profits, where a bidder's gross profit conditional on winning the auction is her valuation of the object θ_i less the amount she pays b_i. Net profits subtract bid costs, which are incurred whether the bidder wins the auction or not.[8]

The order of moves is predetermined, but we will show that the expected profit for the bidder is independent of the order of moves. So the equilibrium we derive also applies with an endogenous order of moves. The first bidder (FB) moves first, and may either pass or submit a bid of b_1 greater than or equal to the minimum bid $\underline{b}$. FB's action is revealed to everyone. The second

[7] Because of agency problems some target management teams do not want to sell the firm even when this maximizes shareholder profits. In the U.S., when management puts the firm 'on the auction block,' management is more likely to be legally compelled to sell the firm to the highest bidder. If instead management does not design an auction structure, in some cases bids still arrive and a spontaneous auction similar to the structure modelled here can result.

[8] We model this as a pure private value auction. However, as long as there is at least some private component, there is an incentive to bid high to signal and intimidate the competing bidder, so effects similar to those modelled here are likely to apply in a setting with correlated valuations.

bidder (SB) then can either make a bid $b_2 \geq b_1$ (or $\underline{b}$, if FB passed) or pass. Any number of passes may occur before bidding begins. The auction ends when a bid is followed by the other bidder passing.

A bidder incurs a cost of γ each time he bids, regardless of whether he ultimately succeeds in buying the object. If he passes, he pays nothing. The quantity γ is a pure transaction cost of bidding, and paying it does not yield any information to the bidder about either his own valuation or those of his competitors.

3. A Discussion of the Ratchet Solution

As discussed earlier, the conclusion conventionally drawn is that, in sequential English auctions, each bidder in turn should submit a bid equal to the previous bid plus the minimum bid increment as long as the resulting bid is less than his valuation. We refer to this outcome as the *ratchet solution*.

When bid costs are zero, it is indeed always a weakly dominant strategy to continue for a bidder to bid as long as the bid level is less than his valuation. (The formal details are unclear in a setting with no minimum bid increment.) To put this another way, since bidding up to one's valuation is weakly dominant, when bid costs are precisely zero, a signalling equilibrium cannot be trembling hand perfect. Thus, the object will be sold to the highest valuation bidder at a price close to the valuation of the second highest bidder.

Since bidding up to one's valuation is weakly dominant, if there is a probability that one's opponent may inadvertently make an incorrect move (*i.e.*, "tremble"; see Selten (1975)), then the weakly dominant strategy is strongly preferred. Even if bidder A has somehow signalled that his valuation exceeds bidder B's, it is still in B's interest to bid until his valuation is reached; A may have trembled to an incorrect signal, or A may mistakenly pass if B bids again. Since B continues bidding until just before his valuation is reached, there is no reason for A to try to signal (by raising the bid by more than the minimum bid increment), which entails a risk of paying more than B's valuation.

In contrast, if the bid cost is strictly positive (i.e., if $\gamma > 0$) ratchet behavior is not weakly dominant.[9] If SB is certain that FB has higher valuation, SB should quit rather than waste his bidding cost. Furthermore, for a given bid cost and with a sufficiently small minimum bid increment, ratchet behavior

[9] Thus, if the likelihood of trembles is small (or infinitesimal, as in the Trembling Hand Perfection equilibrium concept of Selten), the potential expected gain from winning as a result of an opponent's tremble is small, and so is outweighed by even a modest bid cost.

by all bidders ensures negative expected profits for all bidders because the bid cost is incurred many times, and so does not constitute an equilibrium.

Although not the focus of our analysis, when bid costs are low there may exist equilibria that are analogous to the ratchet solution, in which the equilibrium jump in the bid is by a "small" amount that conveys little information about the bidder. High valuation bidders are therefore pooled with low valuation bidders, and successive bids only gradually peel the lowest valuation bidders off of the pool. Such pooling equilibria are wasteful. Learning is slow, so many rounds of bidding cost are incurred, whereas in a signalling equilibrium the bidding ends quickly. Moreover, in a pooling equilibrium there is likely to be an incentive for a high valuation bidder to defect by jumping to a higher bid to signal his type. If this incentive to drive out competitors is stronger for a high valuation bidder than for a low valuation bidder, this defection may credibly reveal high valuation, breaking the pooling equilibrium. Equilibria based on separating behavior by the first bidder are analyzed in the sections that follow.

4. A Signalling Equilibrium with Costless Bidding

We now present an equilibrium with costless bidding in which *FB* makes a high bid which perfectly reveals his valuation. The costless case is useful as a tractable form of the model that lends itself to comparison with the ratchet solution and other costless-bidding auction mechanisms. With zero bid costs, the equilibrium given here is weak: *FB* is indifferent between making the truth revealing bid and any lower bid. Also, as discussed in Section 3, this equilibrium is not trembling hand perfect. However, these weaknesses of the equilibrium obtain only for a bidding cost of exactly zero. In Section 5 we show that when bidding is costly the signalling equilibrium is strong, and that *FB* strictly prefers to make the truth revealing bid. Thus, the behavior described here is best viewed as the limiting case of the equilibrium as positive bidding costs *approach* zero.

For simplicity, in this section we rule out defections that involve passing with the intent of bidding later. This is dealt with in a later section, which shows that such defections do not increase expected profits, and with positive bid costs, strictly reduce expected profits.

We maintain the assumptions outlined in Section 2. We will focus on an equilibrium in which *FB* makes a bid that fully reveals his type, and *SB* responds with either: (1) a bid at $\hat{\theta}_1$, the signalled valuation of *FB*, in which case *FB* will then pass and *SB* will win; or (2) by passing himself, in which case *FB* will win. This signalling equilibrium is weak in the sense that each bidder is indifferent between making his equilibrium bid and any lower bid. In the remainder of this section we state and verify the equilibrium.

4.1 The Proposed Equilibrium

As a solving method we examine *FB*'s decision assuming that he plans to bid once (given an equilibrium response on the part of *SB*). The next subsection verifies that defections involving multiple bids do not increase expected profits. Under the equilibrium conjecture that *FB* plans on making a single bid, *FB* maximizes his expected profit:

$$(\theta_1 - b_1)\, F_2\left(\hat{\theta}_1\,(b_1)\right). \tag{1}$$

FB's gain if he wins the auction is $\theta_1 - b_1$. Since *FB's* bid of b_1 signals his valuation to be $\hat{\theta}_1(b_1)$, based on the equilibrium conjecture that *SB* quits if and only if his valuation is below *FB*'s signalled valuation, *FB*'s probability of winning with this first bid is $F_2\left(\hat{\theta}_1\,(b)\right)$. *SB*'s equilibrium response is to pass if $\theta_2 < \hat{\theta}_1$,[10] and bids $b_2 = \hat{\theta}_1$ and win if $\theta_2 > \hat{\theta}_1$.

We define a *skeptical* inference by *FB* as one in which makes the minimal inference about *SB*'s valuation. *SB*'s equilibrium bid is a direct result of the assumption that, in response to an out-of-equilibrium move by *SB* of bidding less than $\hat{\theta}_1$, *FB* would infer skeptically that *SB*'s bid is virtually as high as his valuation, *i.e.,* that $\hat{\theta}_2(b_2) = b_2$. This inference is based on the conjecture that *SB* bids (very close to) his full valuation (which is no better for *SB* than passing).[11]

Since this is a continuous game, the standard equilibrium refinement concepts are not formally defined. However, this equilibrium satisfies the intuition of standard concepts such as the intuitive criterion of Cho and Kreps (1987).

Given his beliefs, *FB* with valuation $\theta_1 > b_2$ would respond to a bid $b_2 < \hat{\theta}_1 - \gamma$ by bidding (infinitesimally higher than) b_2, expecting that *SB* would be forced to pass. Similarly, *FB* would continue to employ the strategy of upping *SB*'s bid slightly until *SB* passes or until *SB*'s bid equals or

[10] As discussed in Section 3, with zero bidding costs passing is a weakly dominated strategy. However, if there is a positive bidding cost, however small, then after θ_1 is revealed, *SB* with $\theta_2 < \hat{\theta}_1$ strongly prefers to pass. Thus, the equilibrium described here is a limiting case of the strong equilibria when there are positive bidding costs (as in Section 5).

[11] Less extreme skepticism yields essentially the same results. An inference that θ_2 is slightly higher valuation than b_2 still supports the equilibrium. When the inference is $b_2 + \epsilon(b_2)$, where $0 < \epsilon(b_2) < \theta_1 - b_2$, *FB* believes his valuation is higher and hence, with $\epsilon(\cdot)$ sufficiently small, can win with a bid very close to b_2. This would deter *SB*'s defection.

exceeds θ_1. Thus, *SB* cannot gain from a bid below $\hat{\theta}_1(b_1)$, and always follows the equilibrium strategy.

In deriving the signalling schedule, it is convenient to let $b_1(\hat{\theta}_1)$ denote the inverse function $\hat{\theta}_1^{-1}(\cdot)$, *i.e.*, the amount that must be bid to signal a valuation of $\hat{\theta}_1$. The inverse is single-valued under the conjecture, to be verified, that the equilibrium bid is a strictly increasing function of *FB*'s type. *FB* can thus be viewed as maximizing his profits over his signalled valuation,

$$\pi_1(\theta_1) = \max_{\hat{\theta}_1} F_2(\hat{\theta}_1) \left[\theta_1 - b_1(\hat{\theta}_1)\right]. \tag{2}$$

This problem is essentially identical to that solved by a bidder in a static, symmetric first price sealed bid auction (see, *e.g.* Milgrom and Weber (1982), Riley (1989) and Krishna (2009)).[12] However, in our setting the bidders are positioned asymmetrically. Differentiating with respect to $\hat{\theta}_1$ and equating to zero gives the first order condition for the optimal signalled type $\hat{\theta}_1$ gives the global optimum. If bids are fully and truthfully revealing, $\hat{\theta}_1 = \theta_1$, which gives the standard linear first order differential equation for the first price sealed bid auction. Imposing the initial condition that some type θ_1^*, the lowest type to bid, submits a bid of b^* gives the standard unique solution:

$$b_1(\theta_1) = \frac{1}{F_2(\theta_1)} \left(\int_{\theta_1^*}^{\theta_1} s f_2(s) ds + b^* F_2(\theta_1^*) \right) \quad \text{for } \theta_1 > \theta_1^*, \tag{3}$$

If there is a bidding schedule with a discontinuity only at θ_1^*, this solution is still valid for $\theta_1 > \theta_1^*$ if $\lim_{\theta_1 \to \theta_1^{*+}} b(\theta_1) = b^*$. The following lemma shows that the relevant initial condition is $\theta_1^* = b^* = \max(\underline{\theta}, \underline{b})$.

Lemma 1. *If:*

1. *If $\underline{b} > \underline{\theta}$ then $\theta_1^* = \underline{b}$, and $b_1(\underline{b}) = \underline{b}$.*
2. *$\underline{b} \leq \underline{\theta}$, then $\theta_1^* = \underline{\theta}$ and $\lim_{\theta \to \underline{\theta}^+} b_1(\theta_1) = \underline{\theta}$.*

Proof. See appendix.

The intuition for part 1 of this lemma is that, first, $\theta_1^* > \underline{b}$ cannot be an equilibrium: a bidder with valuation $\theta_1 > \underline{b}$ (but less than θ_1^*) could obtain a positive expected profit from bidding $\underline{b}$, versus zero from passing, and would

[12] In both settings, $\hat{\theta}_1$ is the critical value for the opponent's valuation below which the bidder wins the auction, and $b_1(\hat{\theta}_1)$ is the amount a bidder needs to offer to achieve that critical value.

therefore bid. Also, θ_1^* cannot be less than $\underline{b}$, as a bidder with valuation less than $\underline{b}$ would lose money by bidding.

The intuition for part 2 is that as θ approaches $\underline{\theta}$, a limiting bid greater than $\underline{\theta}$ cannot be an equilibrium because a low-type bidder would lose money. Also, a limiting bid less than $\underline{\theta}$ cannot be an equilibrium: a bidder with valuation close to $\underline{\theta}$ would be almost sure to lose, and therefore would have an incentive to bid higher to mimic a higher type. He would thereby obtain a non-negligible expected profit. (This reasoning requires our assumption that $F_i(\theta)$ is twice differentiable, implying that there is no mass-point at $\theta = \bar{\theta}$.)

If Part 1 applies (i.e., if $\underline{b} > \underline{\theta}$) then equation (3) is valid for $\theta_1 = \underline{\theta}$. However, if Part 2 applies (i.e., if $\underline{b} \leq \underline{\theta}$) then, since $F_2(\underline{\theta}) = 0$, the bid schedule (3) is not defined at $\theta_1 = \underline{\theta}$ because the second term of the integrand in equation (3) is zero. The lowest type $\underline{\theta}$ has a zero probability of winning in a revealing equilibrium, so he is equally well off with a bid of $\underline{b}$, $\underline{\theta}$, or any other bid.[13] In contrast, if $\underline{b} > \underline{\theta}$, then $\theta_1^* = \underline{b}$, and the bid schedule (3) is defined at $\theta = \underline{b}$.

With these initial conditions, letting $\underline{x} \equiv \max\{\underline{\theta}, \underline{b}\}$ be the minimum bid that is ever made, FB passes if $\theta_1 < \underline{b}$, and otherwise bids

$$\begin{aligned} b_1(\theta_1) &= \begin{cases} \frac{1}{F_2(\theta_1)} \left[\int_{\underline{x}}^{\theta_1} s f_2(s) ds + \underline{x} F_2(\underline{x}) \right] & \text{if } \theta_1 > \underline{x} \\ \underline{x} & \text{if } \theta_1 \leq \underline{x} \end{cases} \\ &= E\left[\tilde{x} | \tilde{\theta}_2 \leq \theta_1 \right], \end{aligned} \tag{4}$$

where $\tilde{x}$ is defined by

$$\tilde{x} \equiv \begin{cases} \theta_2 \text{ if } \theta_2 \geq \underline{b} \\ \underline{b} \ \text{ if } \theta_2 < \underline{b}. \end{cases}$$

If $\underline{b} < \underline{\theta}$, meaning that the minimum bid is not a constraint, then $E\left[\tilde{x}|\tilde{\theta}_2 \leq \theta_1\right] = E\left[\tilde{\theta}_2|\tilde{\theta}_2 \leq \theta_1\right]$.[14] In either case, the bidding schedule in equation (4) requires the bid to be equal to what the bidder would on average pay if he wins, according to the ratchet solution in an English auction with a minimum bid of $\underline{b}$.

[13] Surprisingly, the most relevant equilibrium has a discontinuous bid schedule in which the lowest type bids $\underline{b} < \underline{\theta}$. We will see in Section 5 that when bidding costs are positive (however small), the lowest type FB to make an offer bids $\underline{b}$ (*not* $\underline{\theta}$), and the bidding schedule is continuous. In the limit as the bidding costs go to zero, the bidding schedule converges pointwise to the schedule just described (with a discontinuity at $\underline{\theta}$). (Since the limit function is discontinuous, the convergence is not uniform.).

[14] SB's bid schedule can also be viewed as taking the form of equation (4) with 1's and 2's reversed, and with a minimum bid of θ_1. That is, SB will bid only if $\theta_2 > \theta_1$, and the bid will be $b_2(\theta_2) = E[\theta_1|\theta_1 < \theta_2] = \theta_1$ since *FB's* valuation is known at the time of SB's decision.

Since the optimization problem is equivalent to that in a standard first price sealed bid auction, the first order condition gives the global optimum. This expected profit is the same as in other optimal auctions, and is the same as under the ratchet solution. Further, *SB*'s expected profit is the same as *FB*'s. Thus both players are indifferent between this equilibrium and the ratchet solution, and are indifferent between bidding first and second. Thus, the standard bidder-profit-equivalence expected profit function for optimal static auctions obtains for both *FB* and *SB*. We note that bidders' profits and bid schedules can also be derived using an envelope condition argument (see, *e.g.*, Milgrom and Weber, 1982).

Thus, with zero bid costs the truth-revealing equilibrium in the sequential bidding auction provides identical bidder expected profits and seller expected revenue to those of other well-known efficient auctions. Since in equilibrium the bidder with highest valuation wins, this follows by standard revenue equivalence reasoning (see, e.g., Milgrom and Weber (1982), Riley (1989), and Krishna (2009)). These revenue and profit equivalences also hold in the setting with positive bid cost in the limit as the bid cost approaches zero.

4.2 Weak Optimality of the Proposed Equilibrium

The previous section showed that *FB* does at least as well bidding according to the schedule given in (4) as making a defection in which only a single bid is planned (given equilibrium behavior by *SB*). To verify that this solution is an equilibrium, we also need to show that it is unprofitable for *FB* to defect by (1) making an initial low bid b_1' such that $\hat{\theta}_1(b_1') < \theta_1$, and then (2) bidding a second time if *SB* does not then pass. With such a 'low-bid' strategy *FB* could pay a lower price for the item if *SB* passes, but could still potentially win the auction by rebidding, if *SB* bids.[15] However, we show in the appendix that such a strategy provides exactly the same expected profit. (In the setting with positive bidding costs, a low-bid deviation yields strictly lower profits.) Specifically, in a dynamic defection strategy by *FB* that ultimately signals value, the total expected profits are equal to that obtained with only a single bid. Any such defection that does not ultimately signal value accurately generates strictly lower expected profits. Thus, no defection generates expected total profits greater than the proposed equilibrium.

[15] *FB* would never bid above the proposed equilibrium bid level. If *FB* were to do so, the only equilibrium bid *SB* can make in response is $b_2 = \hat{\theta}_1(b_1) > \theta_1$. *FB* would never respond to this bid since it exceeds valuation. But if *FB* plans only a single bid, then we have shown that the bid level given in (4) is optimal.

Proposition 1. *If there are two risk neutral bidders who can bid costlessly, then there exists a weak perfect Bayesian equilibrium such that:*

1. FB*'s bid, as shown in equation (4), is equal to what* FB *would on average pay, given that he wins, in a ratchet solution.*
2. SB*, if he wins, pays* $\max\{\underline{b}, \theta_1\}$*, exactly what he pays in a ratchet solution.*
3. *Based on 1. and 2., both bidders and the seller are indifferent between the signalling equilibrium and the ratchet solution, given risk neutrality on the part of all three. If the seller is risk-averse, he prefers the signalling equilibrium.*
4. *Since the expected profit conditional on valuation θ_i is equal for a first and second bidder, bidders are indifferent between moving first and second.*
5. *The probability that* SB *makes a bid is decreasing with the level of the first bid.*

Proof. See appendix.

The evidence of Jennings and Mazzeo (1993) that a high takeover bid premium is associated with a lower probability of competing offers, is consistent with implication 5. The model has a further empirical implication that the jump in the second bid is a increasing function of the first jump:

Proposition 2. *The jump between the first and the second bid, $\theta_1 - b_1$ is increasing in the initial bid b_1 for all identically distributed valuation densities f such that $b_1'(\theta) < 1$, which includes all distributions that satisfy log concavity of the density f or the distribution F.*[16]

Proof. The difference between the first and the second bid can be written as $\theta_1 - b_1 = \theta_1 - E[\theta_2|\theta_2 < \theta_1]$. This object, as a function of θ_1, is called the *mean-advantage-over-inferiors* of a distribution. The conclusion therefore follows so long as this object is increasing in θ_1. Log-concavity of f or F is a sufficient condition (Bagnoli and Bergstrom, 2005).

[16] From equation (3), $\lim_{\theta\to\theta^*,+} b_1(\theta) = \theta^*$; and since the highest valuation bidder makes positive expected profit, $b_1(\overline{\theta}) < \theta$. So averaging b_1' over the range $[\theta^*, \overline{\theta}]$ shows that

$$\frac{1}{\overline{\theta} - \theta^*} \int_{\theta^*}^{\overline{\theta}} b_1'(s) ds = \frac{b_1(\overline{\theta}) - b_1(\theta^*)}{\overline{\theta} - \theta^*} < 1.$$

To prove that the bid jump size $\theta - b(\theta)$ is monotonically increasing in $b(\theta)$ we describe conditions under which which $b'(\theta) < 1$ is true for all θ, not just on average.

Most standard distributions satisfy log concavity. The implication that the jump between the first and second bid is increasing in b_1 has not, to our knowledge, been tested.

5. The Equilibrium with Positive Bidding Costs

The previous section demonstrated that, with zero bidding costs, there is a weak perfect Bayesian equilibrium in which the first bidder (FB) makes an initial, fully revealing bid. If the second bidder (SB) has the higher valuation, then the second bidder responds with a bid of θ_1 and wins the auction. If the second bidder has the lower valuation, then second bidder passes and that first bidder wins the auction. The object is allocated to the highest valuation bidder with a maximum of two bids.

With zero costs, this is a weak Bayesian equilibrium in the sense that FB is indifferent between the equilibrium bid and any lower bid. With positive bidding costs, we now show that the equilibrium behavior becomes strongly optimal. Intuitively the single-bid equilibrium becomes strong because, if the bidder deviates from the equilibrium bidding strategy, this will result in multiple rounds of bidding. If bidding costs are zero, this doesn't affect the payoff to the bidders. However, when bidding costs are positive, the payoff to the bidders are lower if they deviate from the equilibrium strategy because they have to pay the bidding costs multiple times, driving down their expected gains.

Our procedure to derive an equilibrium will be similar to that of Section 4: First, we derive the equilibrium bidding schedules assuming single-bid strategies, *i.e.,* that FB plans to bid only once (if the other bidder follows his equilibrium strategy). Then, we examine *FB's* general maximization problem, contemplating multiple-bid defections, to verify that this single-bid strategy is strictly optimal even when multiple bids are allowed.

A key distinction between the equilibrium in the zero cost setting explored in Section 4 and setting here in which bidding is costly, is that in this setting there can be delays in bidding, where FB initially passes, but later bids. Recall that the assumption here is the cost associated with making a bid is γ, but a “pass” is costless. Delays occur because FBs with valuations close to $\underline{\theta}$ will not bid initially, because the potential gain from bidding is lower than the bidding cost γ that must be paid. However, if following FB's pass SB also passes, FB then correctly infers that SB's valuation is below some threshold, thus increasing FB's assessment of his probability of winning. At this point, if FB's valuation is not too low, he will then enter the bidding.

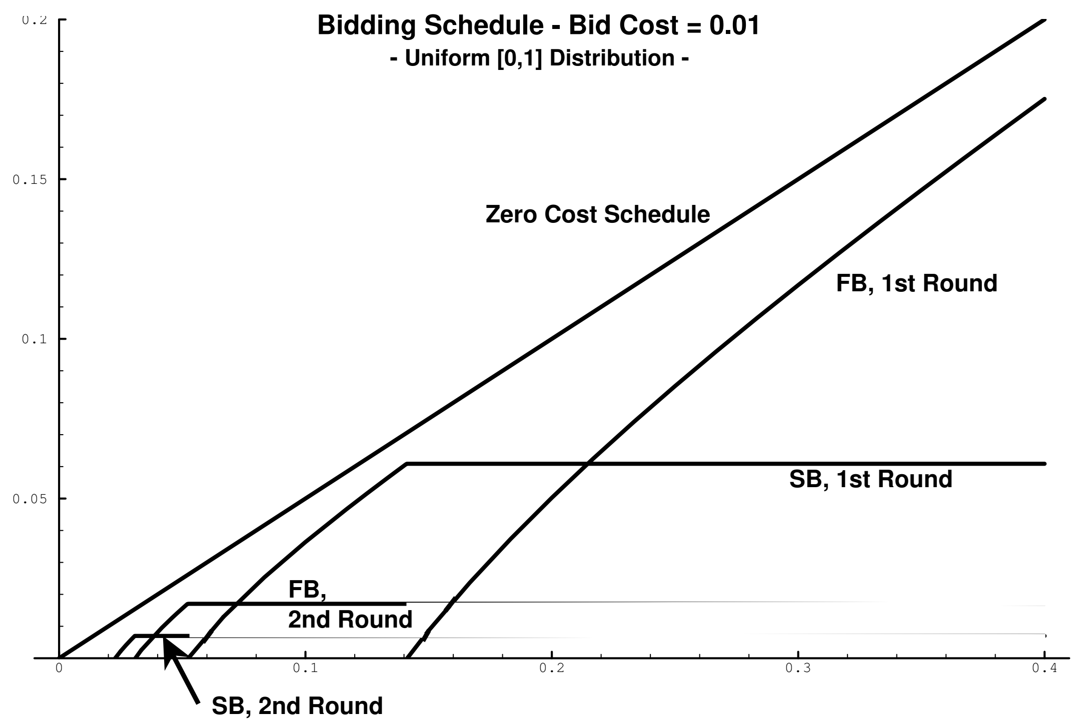

Fig. 1. First Four Bidding Schedules, Uniform $[0, 1]$ Example, $\gamma = 0.01$

To illustrate the equilibrium in the setting with positive bidding costs, consider a setting where the minimum bid $\underline{b} = 0$, where both bidder's valuations are drawn from a uniform distribution on $[0, 1]$, and where the bidding cost $\gamma = 0.01$. In the first round of bidding, a FB with a valuation of $\theta_1 = 0.1$ will clearly choose to pass rather than bid: the reason is that his unconditional probability of having the highest valuation is only 0.1, and his profit if he wins with a bid of $\underline{b} = 0$, is 0.1. Thus, his expected profit, net of bid cost is $(0.1 \cdot 0.1) - 0.01 = 0$. On the other hand, if he passes now, he may have a chance to bid and earn positive profits in the future, providing SB's valuation turns out to be low.[17]

FB will only want to bid if his valuation is greater than a critical value denoted θ_1^*. The equilibrium derived in Subsection 5.1 implies that $\theta_1^* = 0.14$ in this example. If $\theta_1 > \theta_1^*$, from equation (3), with the boundary condition

[17] Delay is encouraged by our assumption that there is no cost of passing with the intent of bidding later. Realistically, such delay is probably costly. However, as discussed in the introduction, in many applications it seems likely that the cost of bidding exceeds the cost of delaying one move. Under such a modeling assumption, there would still be an incentive for a low valuation bidder to delay. A possible further model extension would be to allow for costless quitting, as distinct from costly delay.

that θ_1^* is equal to *FB*'s bidding cutoff, *FB* will bid according to the schedule shown in Figure 1, and this bid will reveal his valuation. Then, analogous with the analysis in Section 4, if *SB*'s valuation is below *FB*'s signalled valuation ($\theta_2 \leq \hat{\theta}_1$), *SB* will pass and *FB* will win. If $\theta_2 > \hat{\theta}_1$, then *SB* will bid $\hat{\theta}_1 - \gamma$, and *FB* will then pass and *SB* will win.

In the case where $\theta_1 < 0.14$, *FB* will pass. Following a pass by *FB*, *SB* correctly infers that *FB*'s valuation is below 0.14. *SB* then knows he faces a weak opponent and is therefore willing to bid even if θ_2 is considerably lower than 0.14. We will show in Subsection 5.1 that *SB* will bid if $\theta_2 > \theta_2^* = 0.05$, and that he will bid according to the schedule in the Figure labeled "*SB*, 1st Round." *SB* will never bid to signal a valuation higher than θ_1^*; since *SB* knows at this point that $\theta_1 < \theta_1^*$, he can win with certainty by bidding high enough to signal a valuation of θ_1^*. Any higher bid would be wasteful.

Finally, suppose that that both $\theta_1 < 0.14$ and $\theta_2 < \theta_2^*$. Then both *FB* and *SB* will pass in the first round, and *FB* will have a second chance to bid. *SB*'s pass reveals to *FB* that $\theta_2 < \theta_2^* = 0.05$. Knowing that *SB* is very weak, *FB* is now willing to bid with any valuation $\theta_1 > \theta_3^* = 0.03$. If *FB*'s valuation is above this cutoff, he will bid according to the schedule in Figure 1 labeled 'FB, 2nd Round.' Passing can continue indefinitely. We will show later that, as long as one of the bidders valuations is greater than $\underline{b} + \gamma$ (in this case 0.01), a bid will eventually occur, and the highest valuation bidder will win the auction.

In the general analysis, we call the first player who in equilibrium submits a bid *FTB* (*First-To-Bid*), and his opponent *STB* (*Second-To-Bid*). Recall that the auction only ends after a bid is followed by a pass. Let *F* and *S* subscripts designate *FTB* and *STB* respectively. Let an n subscript refer to the n'th move after a sequence of $n-1$ passes, $n \geq 1$ (so an odd n refers to *FB* and an even n to *SB*). Let $b_n(\theta_F)$ be the bid of *FTB* as a function of his valuation when the first bid occurs in the n'th move, and let $\hat{\theta}_n(b_n)$ be the inference by *STB* about *FTB*'s valuation if the first bid is made on the n'th move.

Proposition 3. *If bidding costs are $\gamma > 0$, then there exists a perfect Bayesian equilibrium such that:*

1. *Following a sequence of $n-1$ passes ($n \geq 1$), at move n* FTB *with a valuation of θ_F will submit a bid of $b_n(\theta_F)$ if $\theta_F \geq \theta_n^*$, where θ_n^* is a constant, and pass otherwise.*
2. *Suppose that after a sequence of $n-1$ passes* FTB *submits a bid of $b_n(\theta_F)$. Then* FTB*'s valuation is correctly inferred by* STB *to be θ_F,*

and STB *with valuation* θ_S *will bid* $b_S = \theta_F - \gamma$ *if* $\theta_S > \hat{\theta}_n(b_n)$ *and pass otherwise.*

3. FTB*'s bidding schedule is given by:*

$$b_n(\theta_F) = \begin{cases} \frac{1}{F_S(\theta_F)} \left[\underline{b} F_S(\theta_n^*) + \int_{\theta_n^*}^{\theta_F} t f_S(t) dt\right] & \text{if } \theta_F \le \theta_{n-1}^* \\ \frac{1}{F_S(\theta_{n-1}^*)} \left[\underline{b} F_S(\theta_n^*) + \int_{\theta_n^*}^{\theta_{n-1}^*} t f_S(t) dt\right] & \text{if } \theta_F > \theta_{n-1}^*, \end{cases} \tag{5}$$

where $F_S(\cdot)$ *and* $f_S(\cdot)$ *are the prior distribution and density functions of* STB*'s valuation.*

4. *The inference schedule* $\hat{\theta}_n(b_n)$ *is given by the inverse of the bidding function above for* $b_n \in [\underline{b}, b_n(\theta_{n-1}^*))$.

5. *The sequence of critical valuations* θ_n^*, $n \ge 0$ *has the properties that*
   a. $\theta_{n+1}^* < \theta_n^*$,
   b. $\theta^* \equiv \lim_{n\to\infty} \theta_n^* = \max\{\underline{\theta}, \underline{b} + \gamma\}$.

   The sequence is defined by $\theta_0^* \equiv \overline{\theta}$, *and for* $n > 0$ *the iterative relation:*

$$\gamma \left[F_S(\theta_{n-1}^*) - F_S(\theta^*)\right] = \int_{\theta^*}^{\theta_n^*} (t - \underline{b}) f_S(t) dt. \tag{6}$$

   where $\theta^* \equiv \max\{\underline{\theta}, \underline{b} + \gamma\}$.

6. *If* $\theta_1, \theta_2 < \underline{b} + \gamma$, *the object is not sold. Otherwise, it is sold to the highest valuation bidder.*

This perfect Bayesian equilibrium is supported by the out-of-equilibrium belief that, if at step 2, STB *submits a bid* $b_S < \theta_F - \gamma$, FTB *believes that* STB*'s valuation is (close to)* $b_S + \gamma$, *and consequently revises his bid to just above* b_S.

Our proof of this proposition proceeds as follows. First, in this subsection, we demonstrate that the solution to the differential equation that governs the equilibrium bidding schedule is given by equation (5). Next, in Subsection 5.1, we derive the critical value sequence θ_n^* given in equation (6), which completes the description of the equilibrium bidding schedule.

We then address potential defections from the equilibrium strategy. To demonstrate the robustness of the proposed equilibrium, we need to show three things: (i) given the proposed equilibrium, if $\theta_1 > \gamma + \underline{b}$ or $\theta_2 > \gamma + \underline{b}$, then the object is sold to the highest valuation bidder, and otherwise is not sold (Part 6.); (ii) bidding versus passing is optimal behavior as specified, and (iii) bidding according to the bidding schedule is optimal behavior. Point (i) is straightforward. Based on the rules as specified earlier, at any given point in the game, in this equilibrium a bidder quits only if the other bidder has made an offer sufficient to signal a higher valuation. Therefore, if the object is sold at all, it goes to the high valuation bidder. If neither bidder can generate positive profits from a bid at the minimum bid, because both bidders have valuations less than $\underline{b} + \gamma$, then the object will not be sold. Points (ii) and (iii) will be established in Subsections 5.2 and 5.3, respectively.

We first show that object is sold to the highest valuation bidder, and otherwise is not sold.FTB's maximization problem when he bids in a given round is to choose his bid b_F so as to maximize his expected profit given equilibrium response by STB,

$$\max_{b_F} \ (\theta_F - b_F) \, F_S^* \left(\hat{\theta}(b_F) \right) - \gamma, \tag{7}$$

where $F_S^*(\cdot)$ is the distribution function of STB's valuation conditional on FTB's current information, which may include the information that STB has passed in one or more prior bidding rounds. Comparing with the problem in (2), it is clear that precisely the same first order condition applies, leading to the same differential equation with the same solution (equation (3)). When bidding is costly, a lower range of types will not bid because their expected gross profit would be less than the cost of submitting a bid. Thus, on the first move in which a bid occurs, some FTB type $\theta_F^* > \underline{\theta}$ can credibly separate from all lower types by submitting a bid of $\underline{b}$, no matter how low $\underline{b}$ is set. Imposing the boundary condition that θ_n^* be the lowest valuation type to bid and that such a bidder makes the minimum bid $\underline{b}$ gives the upper branch of equation (5). FTB knows that $\theta_S \leq \theta_{n-1}^*$. So when θ_F is so high that $b_n(\theta_F) \geq \theta_{n-1}^*$, FB has signalled as strongly as necessary to drive out STB with certainty. This confirms the 'topping out' of the bid schedule given in the lower branch of of equation (5). This confirms (iii) under the restriction that FTB plans to bid once (given equilibrium behavior by STB). Subsection 5.3 examines general defection strategies involving bidding below the equilibrium bid with the plan of bidding again later.

The lowest valuation FTB to bid on a move must be indifferent between bidding and not; if bidding were strictly preferable, a slightly lower type would have incentive to mimic, as in the proof of Lemma 1. To derive this boundary condition, we must determine the entire sequence of critical values, which we do next.

5.1 Derivation of the Critical Value Sequence

This section derives the equilibrium bid versus pass sequence of critical values. In this equilibrium, the first bidder will bid only if his valuation is above a certain critical value θ_1^*, and will pass otherwise, giving SB the opportunity to make a first bid if $\theta_2 > \theta_2^*$. We calculate the decreasing sequence θ_n^* by solving for the valuation of the bidder who is indifferent between bidding zero and passing in the n'th move, assuming there have been $n-1$ prior passes.

If n is odd, the bidder under consideration is FB, and if n is even, this is SB. This bidder will submit a bid only if his expected profit from doing so is at least as high as his expected profit from passing. Given continuity, a bidder who submits the minimum bid of $\underline{b}$ must be indifferent between bidding and passing. Therefore, to calculate θ_n^*, the type who will submit a bid of $\underline{b}$, we equate the expected profit from bidding $\underline{b}$ to the expected profit from passing.

If the n'th move bidder has valuation θ_n^*, he becomes first to bid by submitting a minimum bid of $\underline{b}$. His expected profit from bidding is his payoff if he wins $(\theta_n^* - \underline{b})$, times the probability that he wins, minus the bid cost γ:

$$\pi_n^B = (\theta_n^* - \underline{b}) \Pr(\theta_S < \theta_n^* | \theta_S < \theta_{n-1}^*) - \gamma = (\theta_n^* - \underline{b}) \left[\frac{F_S(\theta_n^*)}{F_S(\theta_{n-1}^*)} \right] - \gamma. \quad (8)$$

His profit from passing depends on whether his opponent, in the next round, (a) passes, (b) makes a non-top-out bid, or (c) makes a top-out bid.

To derive the critical value sequence, as given in (6) in Part 5 of Proposition 3, we equate the profits from bidding and passing for type θ_n^*. This is done in the Appendix.

5.2 Strong Optimality of Bid-versus-Pass Rule

Next, we need to demonstrate another part of the conjectured equilibrium (see item (ii) just after the statement of Proposition 3), that if a bidder's valuation is above the bidding cutoff for that round of bidding, than it is always optimal to bid rather than pass.

Lemma 2. *It is optimal for an n'th move bidder whose valuation is above the bidding cutoff θ_n^* to bid. A bidder with lower valuation optimally passes.*

Proof. See appendix

5.3 Strong Optimality of Adhering to the Bidding Schedule

The terminology of FTB (first to bid) applies to the bidder who *in equilibrium* bids first, even when we discuss defections in which he does otherwise. FTB can only defect from the equilibrium strategy in three ways: he can pass, and he can bid either above or below the equilibrium bid. Lemma 2 shows that FTB would never wish to defect by passing. Also, it is clear that FTB, with valuation θ_F, would never make a high bid: were he to do so, signalling his valuation as $\hat{\theta} > \theta_F$, STB's only equilibrium responses would be to bid $\hat{\theta} - \gamma$ or pass. In either case, FTB would not bid again, so this defection would necessarily only involve a single bid. However, we have already shown that, if FTB make only one bid, he prefers to make a truth revealing bid.

This leaves us with one possible remaining defection to consider: a 'low-bid' strategy in which FTB makes an initial low bid, and then potentially makes other bids. The following proposition demonstrates that such a defection is suboptimal.

Lemma 3. *It is never optimal for a bidder who is first to bid to bid below the equilibrium bid schedule and then, if STB responds with a bid, bid again so as to truthfully signal his valuation.*

Proof. See appendix

5.4 Delay, Profits, and Asymptotic Efficiency

To further illustrate the properties of the equilibrium, consider the example used at the beginning of this section where $\underline{b} = 0$, and $\theta, \theta \sim U[0,1]$. In this case, the relation in equation (6) simplifies to

$$\theta_1^* = \sqrt{\gamma(2-\gamma)}.$$

So, for $\gamma = 0.2$, $\theta_1^* = 0.6$. Also, by (6),

$$\theta_n^* = \sqrt{\gamma(2\theta_{n-1}^* - \gamma)}.$$

Using this relation, the sequence of bidding critical values is plotted in Figure 2 for $n = 1, 26$. Also, Figure 1 shows the first four bidding schedules for the uniform $[0,1]$ case for $\gamma = 0.01$. These examples suggests that reasonably modest values of γ lead to relatively large effects on bidding schedules and on delay. The introduction suggested that sequential bidding frequently arises as a spontaneous auction mechanism in the absence of formal organization of the market. The following corollary describes the asymptotic optimality of the signalling equilibrium in the costly sequential bidding auction.

Corollary 1.

1. *Taking the limit as $\gamma \to 0$, the equilibrium of the costly sequential bidding auction described in Proposition 3 approaches the zero cost equilibrium of Proposition 1. Thus, for small γ, delays in bidding vanish, the equilibrium is asymptotically efficient, and for a risk averse*

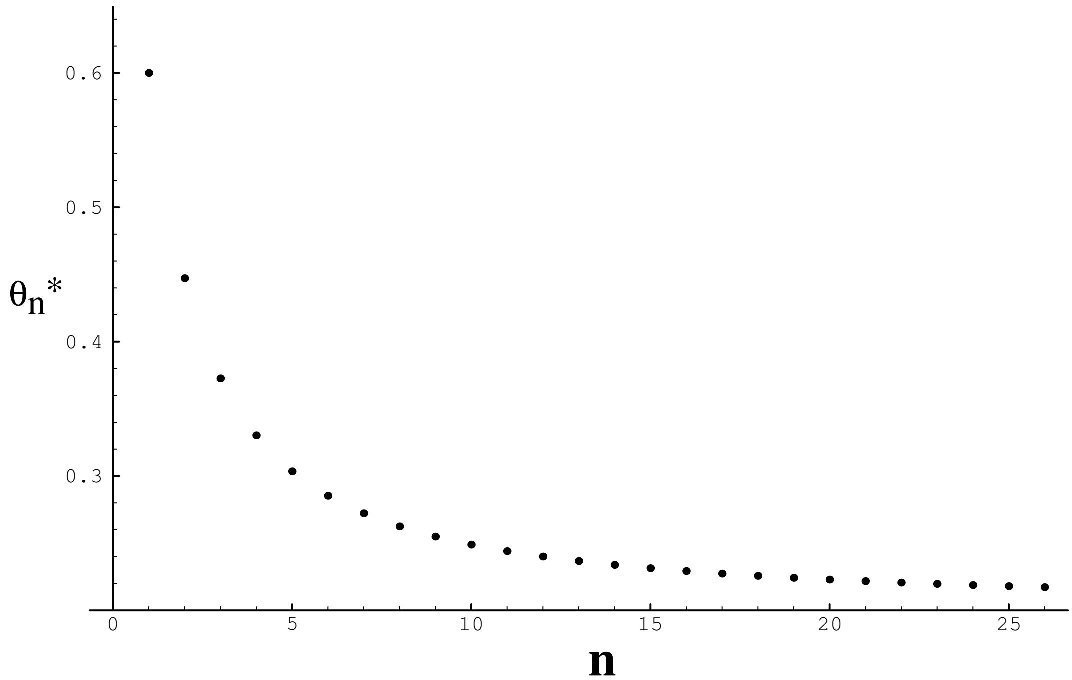

Fig. 2. Bidding Critical Value Sequence, $n = 1, 26$.

seller is preferable to a first price sealed bid (FPSB) auction and the Ratchet Solution.

2. The probabilities of delays in bidding increase monotonically with the cost of bidding.

Proof. Part 1 follows directly from Proposition 3, and by setting $n = 1$ in Part 5(b) and $\theta_1^* = \underline{b}$. Part 2 can be verified by parametrically differentiating (6) in Proposition 3 to show that $d\theta^*/d\gamma > 0$.

Envelope condition techniques can be extended to calculate bidder expected profits. A very tractable expression results.

Proposition 4. FB's *and* SB*'s net expected profits in the skeptical equilibrium of a two-bidder Costly Sequential Bidding auction with bid cost* γ *and minimum bid given* $\underline{b}$ *are*

$$\pi_1(\theta) = \int_{\underline{b}+\gamma}^{\theta} F_2(s)ds \qquad \text{and} \qquad \pi_2(\theta) = \int_{\underline{b}+\gamma}^{\theta} F_1(s)ds. \tag{9}$$

Proof. See appendix.

This is very similar in form to past 'revenue equivalence' auction results. It is surprising to find such a similar form in an asymmetric auction involving dynamic stochastic learning and delay, when bidders incur (stochastic) deadweight bid costs, use by a given bidder of very different bid schedules under different contingencies, and 'topping out' of bids.

The out-of-equilibrium beliefs that support this equilibrium are consistent with the intuition underlying the Intuitive Criterion of Cho and Kreps (1987) (which is formally defined only in settings with a discrete set of possible types). This leads to an equilibrium in which the object is efficiently allocated to the highest-valuation bidder. However, there exist other equilibria involving more 'credulous' beliefs which share some important general properties demonstrated here, such as jumps in bids as signals of value and asymptotic optimality as bid costs become small. These credulous equilibria involve partial pooling, so that a higher valuation bidder is sometimes bluffed out by a lower valuation bidder. Daniel and Hirshleifer (1998) analyze the equilibria in which *FB* makes a credulous inference on seeing an out-of-equilibrium bid by *SB*; see also Bhattacharyya (1992).

6. Comparisons with Alternative Auction Mechanisms

We next compare profits and bidding strategies in a sequential auction with those of alternative auctions. In the next subsection we compare the profits of bidders and the seller in the Costly Sequential Bidding (CSB) auction to a standard static auction, the First Price Sealed Bid (FPSB) auction, and show that the CSB auction is asymptotically optimal as bidding costs become small. In Subsection 6.2, we compare bid schedules in these auctions.

6.1 Profit Comparisons

In this subsection we show that there is bidder profit equivalence between the CSB auction and the FPSB auction. This conclusion is surprising given the seeming complexity of behavior in the CSB auction equilibrium, and such structural differences in the bidding sequence and the resulting possibilities of signalling and of delay, and stochastic deadweight bid costs.

Corollary 2. FB*'s and* SB*'s net expected profits in the skeptical equilibrium of a two-bidder CSB auction with bid cost γ and minimum bid given $\underline{b}$ are equal to to the expected bidder profits in a two-bidder FPSB auction with a minimum bid $\underline{b}+\gamma$ and zero cost of bidding. The sellers' expected revenue is* *smaller in the CSB auction by the expected bid costs; the expected bid costs are less than* 2γ*.*

Expected bid costs are less than 2γ because the auction ends after at most 2 bids. In either auction the object is allocated to the highest valuation bidder, conditional on that bidder's valuation being higher than $\underline{b}$. This means that the total social value, gross of bid costs, is the same for the two auctions. Net of bid costs, the total social value is lower in the CSB auction by the value of the bid costs (which, by assumption, are paid only in the CSB auction). Finally, since the expected bidder profits are the same (by Proposition 5), the sellers expected revenue is therefore lower by the expected bid costs in the CSB auction.

It is interesting to compare the profits in (9) with those of a FPSB auction with a one-shot entry fee:

Proposition 5. *The expected bidder profits in a CSB auction with bid cost* γ *and minimum bid* $\underline{b}$ *are equal to expected bidder profits in a FPSB auction with minimum bid* $\underline{b}$*, and where the only cost of bidding is an entry fee* $e < \gamma$ *set so that a potential bidder will bid if and only if his valuation exceeds* $\underline{b} + \gamma$*. The seller's expected revenues differ in the two auctions by the expected bid costs to be incurred.*

In this setting, bidders do equally well in the CSB auction and the FPSB auction. This is because the bidder-profit-equivalence integral starts with $\underline{b} + \gamma$ and runs up to the bidder's valuation in both auctions. In both auctions the object is sold if and only if the maximum valuation exceeds $\underline{b} + \gamma$. Thus, the same profits are realized in the same states of the world in the two auctions. In other words, gross of bid costs incurred, combined bidder-seller expected profits are identical in the two auctions. However, bid costs incurred differ. Since bidder profits (net of bid costs) are the same, seller expected profits must differ by the difference in the expected bid costs incurred. As a result, if bid costs are positive but small, the CSB auction is close to optimal.

We would expect sellers to incur the costs of designing and organizing formal auction schemes, such as sealed bid auctions, to optimize revenues when more valuable objects are involved. Nevertheless, corporate takeovers are usually allowed to proceed spontaneously as natural sequential bidding auctions. Propositions 2 and 5 offer another possible explanation: if bid costs are low, the "spontaneous" sequential bidding auction is approximately optimal.[18]

[18] A related argument on approximate auction optimality is made by Bulow and Klemperer (1996). They show that an auction with one more bidder is generally superior to a

6.2 Bidding Schedule Comparisons

This section will show that the bidding strategy of *FB* but not *SB* is identical to that of a sealed bid first price auction with an entry fee $e > \gamma$, where each bidder gets to privately observe his valuation before deciding whether to incur the entry fee. As discussed in earlier sections, the differential equation for *FB's* bid in the CSB auction is the same as in the FPSB auction. Therefore, the bidding strategy for the *FB* in the CSB auction will be the same as for the bidders in FPSB if the initial conditions are matched.

The lowest type *FB* in the CSB auction which submits a bid in the first round (θ_1^*) necessarily earns a strictly positive expected profit. Intuitively, an *FB* with valuation θ_1^* must be indifferent between bidding and passing, and his opportunity cost of bidding in the first round is his positive expected profit from passing, with the option to bid later (which is positive). In contrast, the lowest type in the FPSB auction earns 0 if he doesn't bid. Thus, given equal minimum bids in the two auctions, the bidding cutoff is lower in the FPSB auction than for *FB* in the CSB auction. It follows that in order to have the same critical cutoff in the two auctions, the entry fee in the FPSB auction must be higher than the bid cost in the CSB auction. This discussion implies the following result.

Proposition 6. *For any two-bidder Costly Sequential Bidding auction with bid cost γ, there exists a two-bidder First Price Sealed Bid auction with the same minimum bid $\underline{b}$ and with an entry fee $e > \gamma$ such that* FB's *bid schedule in the CSB auction is identical to each of the bidders' schedules in the FPSB auction.*

For several reasons, bidder expected profits and seller expected revenues are not equivalent in these two auctions. *SB*'s response in the CSB auction is to match *FB*'s revealed type; this differs from the bidders' behavior in the FPSB auction. Furthermore, the bid costs differ from the entry fee, and different numbers of individuals may incur these expenses. Also, *FB* obtains profits from later rounds of the CSB auction while the FPSB auction is one-shot. From a profit standpoint, the analogy between the two auctions becomes closer if the FPSB auction can be reopened repeatedly by the seller in the event that neither of the buyers chooses to incur the entry fee. In that case, just as in the CSB auction, waiting will reveal low valuations, so that eventually bids will be made so long as at least one indivdual's valuation

negotiation, suggesting that "it is more worthwhile for a seller to devote resources to expanding the market than to collecting information and making the calculations required to figure out the best mechanism."

exceeds the entry fee. Since the highest valuation bidder will eventually win this auction, we conjecture that profit reasoning implies that bidder profits are the same in the FPSB and CSB auctions when $e = \gamma$, but that expected seller revenue will be higher, and bid costs are usually paid by both bidders in the FPSB auctions. In constrast, in the CSB auction, the bid costs are only paid by *FB*, if *FB* is the higher valuation bidder. In this sense, the CSB auction is more efficient when bidding is costly.

7. Relation to the Existing Literature

Several papers on takeover bidding have recognized the empirical importance of jumps. These papers modify the ratchet solution to allow for a single jump bid. These models therefore have a substantial probability of a large (infinite) number of minimally informative bids. In contrast, our approach implies that there will never be more than a few informative bids. Furthermore, our approach allows for possible delay before bidding commences.

Fishman (1988) examines a setting in which bidding is costless, but the second bidder must pay an initial investigation cost to learn its valuation of the acquisition target. If the investigation cost is substantial, then depending on the first bidder's valuation he may make a substantial jump bid to deter the second bidder from investigating and entering the contest, or may make a lower bid in which case the ratchet solution ensues. However, empirically, jumps in bidding even after the initial bid are common and substantial in takeover contests (Betton and Eckbo, 2000). Also, in Fishman (1988), if the investigation cost is small, then jump bidding disappears, and only one bidder ever signals. In contrast, in our approach even a small bidding cost creates an incentive for all parties to signal with substantial jumps in bids.

Hirshleifer and Png (1990) examine a model of sequential bidding with both investigation costs and costs of bidding in a three-type example to examine the desirability of facilitating competition in takeover contests. The current paper achieves greater tractability by focusing on costs of bidding when investigation is costless.

Bhattacharyya (1992) provides models of initial jump bids based on either an initial investigation cost or a one-time uninformative entry fee. If the second bidder enters then, like Fishman, Bhattacharyya assumes that the target firm is sold according to the ratchet solution.

A rather different approach to jumps in bidding is provided by Avery (1998) in a setting with common rather than private valuations. Avery examines a stylized binary setting where the bidders are allowed to submit only one of two bids, which exogenously consist of either a small or a large increment. Avery shows that, in some cases, the high increment is chosen.[19]

Hörner and Sahuguet (2007) identify an additional source of jump bidding that applies in a different kind of auction mechanism. A bidder may bid low to mislead lull competitors into thinking valuation is low. The reason this is useful is that in the auction considered, there is a second round of bidding with a sealed bid auction determining the winner. A low bid can fool a competitor into bidding low in the second round.

Several recent papers examine alternative forms of strategic interaction in takeover auction models. Chowdhry and Nanda (1993) examine the role of debt as a commitment to bid aggressively. Burkart (1995) investigates a private value auction/takeover setting, and shows that a toehold can induce a potential bidder to bid more aggressively, and can lead to inefficient allocation of the object; see also Singh (1998), Ravid and Spiegel (1999), and the empirical tests of Betton, Eckbo, and Thorburn (2009). In a common value setting, Bulow, Huang, and Klemperer (1999) show that the initial shareholder bids more aggressively, and as a result competitors bid less aggressively.

Povel and Singh (2006) models optimal auction design for target firms when potential bidders are ex ante asymmetric. In consequence, a sequential auction procedure can bring a higher expected selling price than the standard auction. In contrast, in our setting, with bidding costs low, a sequential auction and a simultaneous auction both match the seller with the bidder with highest value. The sequential auction is optimal as it economizes on bidding costs.

8. Conclusion

The traditional ratchet solution for sequential ascending auctions predicts that bidding will always proceed by minimal increments. This minimizes the rate at which bidders learn about each others' valuations, and maximizes the number of rounds of bidding that take place before the auction ends. But what if a cost must be paid each time a bid is submitted or revised? We identify an alternative equilibrium in which, even if bid costs are zero,

[19] In practical applications (such as, e.g., takeover contests) it is almost always possible to bid in very small increments, such as a one-cent versus two-cents. But an explanation for two-cent increments does not explain why a bid at a 20% premium would occur. In Avery's model, the initial bidding serves as a means of selecting and coordinating upon the asymmetric equilibrium to be employed in a later round of bidding. Other authors have examined bids as signals in settings with repeated sales. Bikhchandani (1988) examines reputation effects in repeated auctions and Bikhchandani and Huang (1989) provides a signalling model for auctions of objects with resale markets.

each bidder bids a substantial increment above the minimum or preceding bid, and in which the bid reveals the bidder's valuation. This equilibrium maximizes the rate of relevant learning, so that the each bidder needs to bid at most once.

A high valuation bidder makes a high bid that signals that he is willing to bid even higher for the object. This intimidates a competitor with lower valuation into quitting. A competitor with an even higher valuation jump again, which signals a valuation so high that the initial bidder quits. Even with zero bid costs, this is a (weak) equilibrium. For any positive bid cost, however small, this equilibrium is strong.

The structure of the equilibria relies on the out-of-equilibrium beliefs of the bidders. We focus skeptical beliefs (the lowest possible equilibrium inference by a bidder about the valuation of the competitor), which imply that in equilibrium the higher valuation bidder always wins. The model implies empirically that bidding proceeds in jumps, that the probability of a competing bid decreases with the initial bid, that high initial bids are associated with high subsequent jumps, that delay before the start of bidding is increasing with bid costs, that delays are bad news for the seller (lower expected bids and sale price), and that objects are sold to the highest valuation bidder. From a policy viewpoint, it also implies that even when bid costs are substantial, an ascending bid auction will still be optimal.

The maximized learning in the equilibrium we identify is at the opposite extreme from the ratchet solution. In reality, bidders often bid repeatedly. We conjecture that if a bidder receives progressively more precise signals of his valuation (either exogenously, or based on a choice to investigate more precisely before submitting higher bids), he will sometimes revise his bid upwards to signal the receipt of new favorable information.[20] So even if each bid reveals the bidder's best estimate of his valuation, bidders will sometimes make several bids, each at a significant premium over their opponent's previous bid.[21]

Owing to bid costs, the model implies that bidders may delay an arbitrarily large number of rounds before the first bid is made. Delay reveals that a

[20] For example, further information sometimes arrives during takeover contests, as when a target makes value-relevant disclosures. Also, a target may take defensive actions such as "scorched earth" strategies which affect valuations. Furthermore, a contest can stimulate analysts to generate information about the target and potential synergies.

[21] A related extension could potentially explain why a bidder may sometimes jump above his own previous bid in the absence of an intervening bid (consistent with auction evidence of Cramton (1997)). Self-jumps of this type are by assumption impossible in our model since the auction would end before such an opportunity arose. However, in a setting that matched the PCS auction more closely, the arrival of further favorable information could create a further incentive to signal, and hence a self-jump.

bidder believes he is unlikely to win, and therefore does not find it worthwhile to incur the bid cost. Such a bidder will sometimes gain confidence about his chance of winning after seeing his competitor pass. Eventually, so long as there is a bidder with positive valuation net of the bid cost, bidding must eventually commence. The bidding schedule followed by the first bidder is the same as in a static first-price sealed bid auction in which the bidder must pay an entry fee in order to submit an offer.

Bidding schedules, profits, and seller revenues in the costly sequential bidding auction are related to those of traditional static auctions. When bidding is costly, the first bidder's bidding schedule as a function of his valuation is identical to that in a *first price* sealed bid auction with identical minimum bid and with an entry fee set above the bid cost. After a first bid, the amount actually paid by a victorious second bidder is identical to that in a *second-price* sealed bid auction.

Somewhat surprisingly, traditional revenue equivalence results can be extended, with slight modifications, to this costly dynamic auction. When bidding is costless, bidder profits and seller revenue are the same as in the ratchet solution and as in a First Price Sealed Bid (FPSB) auction. Bidders' net expected profits in the costly sequential bidding auction have the same form as profits in a FPSB auction with an entry fee lower than the bid cost; they are therefore indifferent as to the order of moves.

Furthermore, the costly sequential bidding auction is asymptotically optimal as the bid cost approaches zero. This offers another possible explanation for the persistence of such 'spontaneous auctions' even in markets for high-value objects such as firms. Bulow and Klemperer (2009) find that a simultaneous bidding mechanism is more efficient, but that settings with preemptive bids transfer surplus from the seller to buyers. This in turn encourages bidder entry, and therefore tends to increase seller expected revenue. However, in our setting sequential bidding is asymptotically more efficient (for small costs). Sequential bidding and a simultaneous auction both efficiently allocate the product, but the sequential auction minimizes expected bidding costs.[22]

Several models of takeover auctions have applied the ratchet solution (perhaps with an initial entry or investigation cost) to address such issues as debt as a commitment device in takeovers, the effects of value reducing defensive strategies, and optimal regulation of takeover contests (see, e.g., Chowdhry and Nanda (1993), Povel and Singh (2010), and Berkovitch and Khanna

[22] In Roberts and Sweeting (2013), a sequential mechanism can also be more efficient. In their setting this is driven by an ex ante information asymmetry wherein buyers receive a noisy signal about their valuations prior to the entry decision and acquisition of a more accurate signal. We examine a simpler setting with an exogenous and perfect signal about own-valuation.

(1990)). It will be interesting to see what new implications can be derived applying an auction models where bidding occurs in a series of jumps.

To sum up, in our model, owing to bidding costs, at each opportunity a bidder either passes or jumps to increase the bid by a substantial margin over the previous one. This communicates bidders' information rapidly, leading to contests that are completed with small numbers of bids. This general pattern is consistent with certain types of natural sequential auctions, such as takeover contests. The model provides empirical implications, derives revenue and efficiency relationships between different auctions, describes when players should delay bidding and the information conveyed by such delay. The model also provides distinct implications for a major policy issue: whether simultaneous versus sequential auctions are socially efficient.

1. Appendix - Proof of Propositions

Proof (of Lemma 1).
Part 1: No player will ever bid above his valuation, so $b_1(\theta) \leq \theta$. So $\lim_{\theta \to \underline{\theta}^+} b_1(\theta) \leq \underline{\theta}$. We next show that $\lim_{\theta \to \underline{\theta}^+} b_1(\theta) \geq \underline{\theta}$. Assume to the contrary that $\lim_{\theta \to \underline{\theta}^+} b\,(\theta) < \underline{\theta}$. Then by continuity for any $\epsilon > 0$, however small, $\exists\ \theta' > \underline{\theta}$ such that $b_1(\theta') = \underline{\theta} - \epsilon$. Now, the expected profit of a bidder with valuation $\underline{\theta} + \delta$ $(\delta > 0)$ who makes an equilibrium bid is $F(\underline{\theta} + \delta)[\underline{\theta} + \delta - b_1(\underline{\theta} + \delta)]$, which goes to zero as $\delta \to 0$. However, a mimicking bid of $b_1(\theta')$ yields expected profit $F(\theta')(\epsilon + \delta)$, which remains bounded away from 0 as $\delta \to 0$. Therefore, for sufficiently small δ, a mimicking bid yields higher profit than a bid at the proposed level. This contradicts the assumption, so $\lim_{\theta \to \underline{\theta}^+} b\,(\theta) \geq \underline{\theta}$ Taken together, these two restrictions imply that $\lim_{\theta \to \underline{\theta}^+} b_1(\theta) = \underline{\theta}$.

Part 2: No bidder with valuation below the minimum bid can profit from making an offer. So the lowest type that can break even by bidding is $\theta_1 = \underline{b}$, and this type can break even only if $b_1(\underline{b}) = \underline{b}$. ‖

Proof (of the weak optimality of the equilibrium proposed in Section 4.1).
Suppose that *FB* defects to a low-bid strategy. If *SB* responds with a pass, *FB* wins and pays less than he would have in the proposed equilibrium. However, if *SB* responds with a bid, we show in this section that *FB* pays more if he wins on a second (or later) bid. These effects offset, so his expected profit is equal with the low-bid strategy or with the proposed equilibrium strategy. This confirms the proposed behavior as a weak perfect Bayesian

equilibrium. In Section 5 we show that when bidding is costly, his profit with the low-bid defection is strictly lower, so the proposed equilibrium is strong.

FB's expected profit from the low-bid strategy will depend on *SB*'s inference on seeing the off-equilibrium action of a second bid by *FB*. We make the assumption (to be verified as part of an equilibrium) that *SB* interprets such an action as a new and (this time) truthful separating bid.[23] Therefore, to determine *FB's* profit from the low-bid strategy, we examine a defection in which, after an initial low bid and a responding equilibrium bid by *SB*, *FB* bids so as to signal his true valuation as θ_1.[24]

Given *FB*'s first bid of b_1 such that $\hat{\theta}_1(b_1) < \theta_1$, and given that *SB* responds with an equilibrium bid of $b_2 = \hat{\theta}_1(b_1)$, *FB* now knows that $\theta_2 \in \left[b_2, \overline{\theta}\right]$ with distribution $F_2(\theta_2|\theta_2 > b_2)$. *FB*'s problem in signalling his valuation at this stage is isomorphic with the problem on his first bid in the setting of the previous section in which he had only one chance to bid. Thus, the form of the solution for the bid is the same as in (4), with two differences. First, the unconditional distribution and density functions for θ_2 must be replaced with the functions conditional on $\theta_2 \in [b_2, \overline{\theta}]$,

$$F_2(\theta_2|\theta_2 > \hat{\theta}_1) = \frac{F_2(\theta_2) - F_2(\hat{\theta}_1)}{1 - F_2(\hat{\theta}_1)} \text{ and } f_2(\theta_2|\theta_2 > \hat{\theta}_1) = \frac{f_2(\theta_2)}{1 - F_2(\hat{\theta}_1)}. \quad \text{(A1)}$$

Second, we now have the initial condition that the minimum valuation *FB* who would wish to bid again must at least match *SB*'s bid. When bid costs are zero, the lowest type *FB* that would at least match a bid of b_2 has valuation $t = b_2 = \hat{\theta}_1$.

Replacing the distribution and density functions in (4) with those given in (A1), and using the above initial condition gives the schedule for *FB's* second bid as a function of his true valuation θ_1 and the valuation signalled

[23] We make this assumption in order to be tough on our candidate for equilibrium. This type of belief revision on *SB*'s part encourages defection because in effect *FB* is able to say "I deceived you before, but this time I am signalling truthfully" and *SB* will believe him. If instead, once convinced by the first signal that $\theta_1 = \hat{\theta}_1$, *SB* would never increase his assessment of θ_1 (or at most would update it only to γ plus the new bid), *SB* would be less likely to quit, and a defection to a low bid would become even less profitable. The proposed belief is consistent with the concept of perfect Bayesian equilibrium.

[24] Suppose that there is no net gain to bidding low on the first bid with the plan of countering any bid by *SB* with a *second* and revealing bid. Then an iterative argument can be used to show that there will be no gain underbidding on the second bid in the expectation of making a third revealing bid. This is based on the idea that the second round bidding problem (foreseeing a third round of bidding) is similar to the first round bidding problem (foreseeing a second round of bidding). The appendix (Proof of Proposition 1) shows that there is no net gain to any finite or infinite arbitrary sequence of underbidding in many rounds.

by his first round bid:

$$ {}_2b_1\left(\theta_1,\hat{\theta}_1\right)=\frac{\int_{\hat{\theta}_1}^{\theta_1} tf_2(t|t>\hat{\theta}_1)dt}{F_2(\theta_1|\theta_2>\hat{\theta}_1)}=E[\theta_2|\hat{\theta}_1<\theta_2<\theta_1]. $$

FB's expected profit from the two rounds of bidding is therefore

$$ \begin{aligned} \pi_2(\hat{\theta}_1) &= \Pr(\theta_2<\hat{\theta}_1)\left[\theta_1-b_1(\hat{\theta}_1)\right]+\Pr(\hat{\theta}_1<\theta_2<\theta_1)\left[\theta_1-{}_2b_1(\theta_1)\right] \\ &= \Pr(\theta_2<\hat{\theta}_1)\left(\theta_1-E[\theta_2|\theta_2<\hat{\theta}_1]\right) \\ &\qquad + \Pr(\hat{\theta}_1<\theta_2<\theta_1)\left(\theta_1-E[\theta_2|\hat{\theta}_1<\theta_2<\theta_1]\right) \\ &= \Pr(\theta_2<\theta_1)\left(\theta_1-E[\theta_2|\theta_2<\theta_1]\right) \\ &= \Pr(\theta_2<\theta_1)\left[\theta_1-b_1(\theta_1)\right]. \end{aligned} \tag{A2} $$

The expected profit is therefore equal to the expected profit when *FB* contemplates only a single bid. Thus, there is no gain to the low-bid defection. ||

Proof (of Proposition 1). We showed that the optimal bid of a bidder who plans to bid once is

$$ b_1(\theta_1)=E[\theta_2|\underline{\theta}<\theta_2<\theta_1]. $$

Now suppose that *FB* defects to a low-bid strategy. First, *FB* signals a valuation of θ', then a valuation of θ'', and so on until *SB* drops out or until he finally signals a valuation of θ_1, his true valuation. (It is not optimal to quit before signalling one's true valuation.) With such a strategy, *FB's* probability of winning is the same, $F(\theta_1)$. His expected payment is

$$ \begin{aligned} &E[\theta_2|\underline{\theta}<\theta_2<\theta']\cdot F(\theta') + E[\theta_2|\theta'<\theta_2<\theta'']\cdot[F(\theta'')-F(\theta')] \\ &+ E[\theta_2|\theta''<\theta_2<\theta''']\cdot[F(\theta''')-F(\theta'')] + E[\theta_2|\theta'''<\theta_2<\theta_1]\cdot[F(\theta_1)-F(\theta''')] \end{aligned} $$

This sums to

$$ E[\theta_2|\underline{\theta}<\theta_2<\theta_1]\cdot F(\theta_1). $$

Thus, the expected payment is independent of the number of intermediate bids *FB* makes.

The rest of the proposition is proven in the text except for the part 3 claim that the seller prefers the signalling equilibrium. In the event that $\theta_2<\theta_1$, in the signalling equilibrium the seller receives $E[\theta_2|\theta_2<\theta_1]$, and in the Ratchet Solution receives θ_2. Thus, if $\theta_2<\theta_1$, then the Ratchet Solution expected payment is $R=E[\theta_2|\theta_2<\theta_1]+\theta_2-E[\theta_2|\theta_2<\theta_1]=S+\theta_2-E[\theta_2|\theta_2<\theta_1]$.

If $\theta_2 \geq \theta_1$, $S = R$. Let

$$\epsilon \equiv \begin{cases} 0 & \text{if } \theta_2 \geq \theta_1, \\ \theta_2 - E[\theta_2|\theta_2 < \theta_1] & \text{if } \theta_2 < \theta_1. \end{cases}$$

$$\begin{aligned} E[\epsilon|\theta_1] &= E[\epsilon|\theta_2 < \theta_1] \Pr(\theta_2 < \theta_1) + E[\epsilon|\theta_2 \geq \theta_1] \Pr(\theta_2 \geq \theta_1) \\ &= (E[\theta_2|\theta_2 < \theta_1] - E[\theta_2|\theta_2 < \theta_1]) \Pr(\theta_2 < \theta_1) \\ &= 0. \end{aligned}$$

So ϵ is white noise. $\|$

Proof (of Proposition 2). The second bid is equal to *FB*'s valuation. Therefore the jump between the first and the second bid is the difference between *FB*'s valuation and *FB*'s bid. This is increasing with θ_1 if

$$\frac{d\left[\theta_1 - b_1(\theta_1)\right]}{d\theta_1} > 0, \text{ i.e., } \frac{db_1(\theta_1)}{d\theta_1} < 1.$$

Differentiating the bidding schedule in (3) with respect to θ gives

$$\frac{db_1(\theta_1)}{d\theta_1} = \frac{f(\theta_1)}{F(\theta_1)} \left[\theta_1 - b_1(\theta_1)\right]. \tag{A3}$$

The first bidder's expected profit is:

$$\begin{aligned} \Pi_1(\hat{\theta}_1, \theta_1) &= \left[\theta_1 - b(\hat{\theta}_1)\right] F_2(\hat{\theta}_1) \\ &= \theta_1 F_2(\hat{\theta}_1) - \int_{\underline{\theta}}^{\hat{\theta}_1} s f(s) ds. \end{aligned} \tag{A4}$$

Since this is a fully revealing equilibrium, the inferred valuation must equal the true valuation, that is $\hat{\theta}_1 = \theta_1$. Substituting this into (A4) and integrating by parts gives

$$\pi(\theta) = [\theta - b(\theta)] F_2(\theta) = \int_{\underline{\theta}}^{\theta} F_2(s) ds. \tag{A5}$$

and substituting this into equation (A3) gives

$$\frac{db_1(\theta_1)}{d\theta_1} = \theta_1 F(\theta_1) - \int_{\theta^*}^{\theta_1} F(t) dt.$$

Part 1: For f such that $f'(\theta) \leq 0 \; \forall \theta$,

$$\begin{aligned} F(\theta_1) &= \int_{\theta^*}^{\theta_1} f(t) dt \\ &\geq f(\theta_1)(\theta_1 - \theta^*), \quad \text{so} \end{aligned}$$

$$\begin{aligned}\frac{db_1(\theta_1)}{d\theta_1} &= \frac{f(\theta_1)}{F(\theta_1)}\left(\theta_1 - b_1(\theta_1)\right)\\ &< \frac{f(\theta_1)}{F(\theta_1)}\left(\theta_1 - \theta^*\right)\\ &\leq \frac{f(\theta_1)}{f(\theta_1)\left(\theta_1 - \theta^*\right)}\left(\theta_1 - \theta^*\right)\\ &= 1.\end{aligned}$$

Part 2: If $f'(\theta) = k$, then f and F are:

$$f(\theta) = \frac{2(\theta_1 - \theta^*)}{(\overline{\theta} - \theta^*)^2}, \quad F(\theta) = \frac{(\theta_1 - \theta^*)^2}{(\overline{\theta} - \theta^*)^2}.$$

Direct substitution gives

$$\frac{db_1(\theta_1)}{d\theta_1} = \frac{f(\theta_1)}{[F(\theta_1)]^2}\int_{\theta^*}^{\theta_1} F(t)dt = \frac{2}{3}.$$

Part 3: For any density f such that $f''(\theta) < 0\ \forall\theta$,

$$F(\theta_1) > \frac{1}{2}f(\theta_1)(\theta_1 - \theta^*),$$

because the RHS is the area of the right triangle inscribed under the density $f(\theta)$. Substituting the RHS of this expression for $F(\theta_1)$ in equation (A3) above gives:

$$\begin{aligned}\frac{db_1(\theta_1)}{d\theta_1} &< \frac{f(\theta_1)}{\frac{1}{2}f(\theta_1)(\theta_1 - \theta^*)}\left[\theta_1 - b_1(\theta_1)\right]\\ &< \frac{f(\theta_1)}{\frac{1}{2}f(\theta_1)(\theta_1 - \theta^*)}\left[\theta_1 - \frac{1}{2}\left(\theta_1 + \theta^*\right)\right]\\ &< \frac{2}{(\theta_1 - \theta^*)}\left(\frac{\theta_1 - \theta^*}{2}\right)\\ &= 1.\|\end{aligned}$$

Derivation of the critical value sequence for passing (equation (6), in Proposition 3):

We refer to a bidder as the n'th move bidder or FTB if this is his *equilibrium* behavior. The n'th move bidder's profit if he passes is calculated by considering the three possibilities for the other bidder's valuation: (1) $\theta_S \geq \theta_n^*$, (2) $\theta_n^* > \theta_S \geq \theta_{n+1}^*$ and (3) $\theta_S < \theta_{n+1}^*$. In Case 1, the n'th move bidder will lose the auction and pay no bidding cost. In Case 2, the other bidder will submit an equilibrium bid in the next move, signalling his valuation, to which the n'th move bidder, in order to win the auction, will respond with a bid of $\theta_S - \gamma$ and win. In Case 2, the n'th move bidder will also have to pay the bidding cost of γ. In Case 3, the other bidder will pass in the $(n+1)$'th move. Therefore the n'th move 'equilibrium strategy' given his initial defection is to submit a bid signalling that his valuation is greater than or equal to θ^*_{n+1}, the other bidder's maximum possible valuation at this stage. We will omit S subscripts which belong to all density and distribution functions in the proof. Then the n'th move bidder's expected profit from passing in the n'th move is

$$\pi_P = \theta^*_n \frac{F(\theta^*_n)}{F(\theta^*_{n-1})} - \int_{\theta^*_{n+1}}^{\theta^*_n} (t-\gamma)\frac{f(t)}{F(\theta^*_{n-1})}dt$$
$$-\frac{1}{F(\theta^*_{n+1})}\left[\underline{b}F(\theta^*_{n+2}) + \int_{\theta^*_{n+2}}^{\theta^*_{n+1}} t\frac{f(t)}{F(\theta^*_{n+1})}dt\right]\frac{F(\theta^*_{n+1})}{F(\theta^*_{n-1})} - \gamma\frac{F(\theta^*_n)}{F(\theta^*_{n-1})} \quad \text{(A6)}$$

The first term (which comes from Cases 2 and 3) is just FTB's valuation of the object times the probability of winning (which occurs when $\theta_S < \theta^*_n$) conditional on $\theta_S < \theta^*_{n-1}$. The second term is the n'th move bidder's expected payment for the object in Case 2. The third term is the n'th move bidder's bid (in brackets, equation (5) with n replaced by $n+2$) given Case 3, times the probability that this will occur, $\Pr(\theta_S < \theta^*_{n+1} | \theta_S < \theta^*_{n-1})$. In Case 3, the n'th move bidder will signal that his valuation is θ^*_{n+1}, which is the $n+1$'th move bidder's maximum valuation given that he passes a second time. Finally, the fourth term gives the expected bidding costs incurred in the future by the n'th move bidder given that he passes. (He incurs future bid costs if the $n+1$'th move bidder has $\theta_S < \theta^*_n$, the probability of which is conditioned on knowing that $\theta_S < \theta^*_{n-1}$.) Simplifying this expression gives

$$\pi_P = \frac{1}{F(\theta^*_{n-1})}\left[\theta^*_n F(\theta^*_n) - \int_{\theta^*_{n+2}}^{\theta^*_n} tf(t)dt - \gamma F(\theta^*_{n+1}) - \underline{b}F(\theta^*_{n+2})\right]. \quad \text{(A7)}$$

Equating π_B and π_P gives an iterative relation for the critical value levels.

$$\gamma\left[F(\theta^*_{n-1}) - F(\theta^*_{n+1})\right] = \int_{\theta^*_{n+2}}^{\theta^*_n} (t-\underline{b})f(t)dt \quad \text{(A8)}$$

To derive the general equation, we substitute in $n = i, i+2, \ldots, i+2k, \ldots$ into this equation, recalling that, by definition, $\theta^*_0 = \overline{\theta}$, and taking the telescoping sum of the resulting equations:

$$\gamma\left[F(\theta^*_{i-1}) - F(\theta^*_{i+1})\right] = \int_{\theta^*_{i+2}}^{\theta^*_i} (t-\underline{b})f(t)dt$$
$$\gamma\left[F(\theta^*_{i+1}) - F(\theta^*_{i+3})\right] = \int_{\theta^*_{i+4}}^{\theta^*_{i+2}} (t-\underline{b})f(t)dt$$

$$\vdots$$

$$\gamma\left[F(\theta^*_{i+2k-3}) - F(\theta^*_{i+2k-1})\right] = \int_{\theta^*_{i+2k}}^{\theta^*_{i+2k-2}} (t-\underline{b})f(t)dt,$$

which sums to

$$\gamma\left[F(\theta^*_{i-1}) - F(\theta^*_{i+2k-1})\right] = \int_{\theta^*_{i+2k}}^{\theta^*_{i}} (t-\underline{b})f(t)dt. \tag{A9}$$

The θ^*_n's are monotonically decreasing, because the more passes there have been, the lower a bidder's valuation has to be to make him indifferent between bidding and not. To simplify, we define $\theta^* \equiv \max\{\underline{\theta}, \underline{b}+\gamma\}$ and show that $\lim_{n\to\infty}\theta^*_n = \theta^*$. To show this, we consider two cases. In Case I, $\underline{\theta} < \underline{b}+\gamma$. In case II, $\underline{\theta} \geq \underline{b}+\gamma$.

In Case I, we need to show that $\theta^* \equiv \lim_{n\to\infty}\theta^*_n = \underline{b}+\gamma$. Suppose instead that, in equilibrium, $\theta^* > \underline{b}+\gamma$. Let $\delta \equiv (1/2)[\theta^* - (\underline{b}+\gamma)]$, which is positive in Case I. A bidder with valuation $\theta = \theta^* - \delta$ who makes a first bid of $\underline{b}$ at some move will make a positive profit of $\theta^* - \delta - \underline{b} - \gamma$ with positive probability, as opposed to zero profits according to the equilibrium strategy (always passing). Thus, a bidder with valuation $\theta^* - \delta < \underline{b}+\gamma$ will eventually bid. This contradicts the premise that the limit of the critical values $\theta^* > \underline{b}+\gamma$. Therefore, in Case I, $\theta^* = \underline{b}+\gamma$.

Next, suppose that $\theta^* < \underline{b}+\gamma$, so that $\delta < 0$. The profit of a bidder with $\theta = \theta^* - \delta$ who at any round makes the minimum bid of $\underline{b}$ and wins will be $(\theta^* - \delta) - \underline{b} - \gamma < 0$, as opposed to zero profits if he always passes. Thus, a bidder with valuation $\theta^* - \delta > \theta^*$ will never bid. This contradicts the premise that $\theta^* < \underline{b}+\gamma$. Therefore, in Case I, $\theta^* = \underline{b}+\gamma$.

Similar arguments for Case II shows $\theta^* = \underline{\theta}$ is the only critical value consistent with equilibrium.

Thus, taking the limit of (A9) as $k \to \infty$ gives the iterative definition of the θ^*_i sequence given in (6) in Proposition 3. (This applies to θ^*_1 as well since we define $\theta^*_0 \equiv \bar{\theta}$.) ‖

Proof (of Lemma 2). The terminology of FTB (first to bid) applies to the bidder who *in equilibrium* bids first, even when we discuss defections in which he does otherwise. We show that bidding is strongly optimal if $\theta_F > \theta^*_F$. Specifically, the expected payoff for FTB with $\theta_F > \theta^*_F$ is strictly higher if he follows an equilibrium strategy of bidding in move n than if he defects by passing, with the plan of bidding again in move $n+1$.

Consider the decision of FTB with $\theta_F > \theta^*_n$ at move n given that $n-1$ prior passes have occurred. FTB believes that the other bidder's valuation is below θ^*_n, since STB did not bid previously. The conditional distribution for

STB's valuation is therefore

$$G_S(\theta) \equiv \Pr(t < \theta | t < \theta^*_{n-1}) = \frac{F_S(\theta)}{F_S(\theta^*_n)}.$$

Bidders' profits and bid schedules can be written using an envelope condition argument (see, *e.g.*, Milgrom and Weber (1982)). FTB's move-n optimized expected profits, analogous to equation (2), are

$$\pi^n_F(\theta_F) = \max_{\hat{\theta}_F} G_S(\hat{\theta}_F)\left[\theta_F - b_F(\hat{\theta}_F)\right] - \gamma. \tag{A10}$$

Using the envelope theorem and assuming that the player with the higher valuation always wins the auction, the derivative of (A10) evaluated at $\hat{\theta}_F = \theta_F$ is

$$\frac{d\pi(\theta_F)}{d\theta_F} = G_S(\theta_F).$$

In the region where FTB makes a 'topout' bid high enough to win with certainty, he is at a corner solution, so the envelope theorem does not apply. However, by expression A10, as θ goes up a unit profit goes up by $G_S = 1$ units. Hence, integrating up G_S to generate the profit function gives the correct answer even in this range.

Integrating, the equilibrium expected profit for FTB with $\theta_F > \theta^*_F$ who follows the Equilibrium strategy of bidding at move n is:

$$\pi^E_F(\theta_F) = \pi^E_F(\theta^*_F) + \int_{\theta^*_F}^{\theta_F} G_S(t)dt, \tag{A11}$$

where the first term on the RHS is the expected profit of the lowest type to bid.

Now, define

$$H_S(\theta_F) \equiv \Pr(t < \theta_F | t > \theta^*_F) = \frac{G_S(\theta_F) - G_S(\theta^*_F)}{1 - G_S(\theta^*_F)}.$$

Using the envelope condition, FTB's expected profit from Defecting by passing at move n, and then bidding to maximize expected profits in the next round is:

$$\begin{aligned}\pi^D_F(\theta_F) &= \pi^D_F(\theta^*_F) + G_S(\theta^*_F)(\theta_F - \theta^*_F) + [1 - G_S(\theta^*_F)]\int_{\theta^*_P}^{\theta_F} H_S(t)dt \\ &= \pi^E_F(\theta^*_F) + G_S(\theta^*_F)(\theta_F - \theta^*_F) + \int_{\theta^*_P}^{\theta_F}\left[G_S(t) - G_S(\theta^*_F)\right]dt. \end{aligned} \tag{A12}$$

The first term on the RHS, $\pi^D_F(\theta^*_F)$, is the expected profit of a bidder with $\theta_F = \theta^*_F$ who passes in the n'th move. Since type θ^*_F is indifferent to passing and bidding, this is the same as $\pi_F^E(\theta_F^*)$ in the RHS of (A11). The next two terms come from analyzing the profit of FTB with valuation $\theta_F > \theta_F^*$ relative to FTB with valuation $\theta_F = \theta_F^*$.

After FTB makes an out-of-equilibrium pass, there two possible STB responses to consider. First, if $\theta_S < \theta_F^*$, which happens with probability $G_S(\theta_F^*)$, then STB will either pass or signal his true valuation. In either case FTB will win the auction and will obtain an extra profit $\theta_F - \theta_F^*$ above what FTB with $\theta_F = \theta_F^*$ would earn, which gives the second term.[25]

Second, if $\theta_S \geq \theta_F^*$, which happens with probability $[1 - G_S(\theta_F^*)]$, then STB will make a 'top-out' bid, signalling a valuation of $\theta_S \geq \theta_F^*$. Then FTB may want to bid again. However, if FTB's valuation is only slightly greater than θ_F^*, then he is almost sure to lose to a STB who has signalled valuation $\theta_S \geq \theta_F^*$. The net extra payoff for FTB of this type from making a second bid is negative, so he will pass. Thus it is only a FTB with valuation $\theta_F \geq \theta_P^* > \theta_F^*$ who will want to bid. Moreover, for $\theta_F = \theta_P^*$, the net incremental payoff from making a second bid must be zero. The envelope condition gives FTB's bidder's profit as an integral of the conditional distribution function for STB's valuation (H_S). This gives the third term of (A12).

Taking the difference between the profit from the bidding and passing strategies gives

$$\pi_F^E(\theta) - \pi_F^D(\theta) = \int_{\theta_F^*}^{\theta_P^*} [G_S(s) - G_S(\theta_F^*)]\, ds.$$

Since $\theta_P^* > \theta_F^*$, this quantity is positive, so a bidder with $\theta_F > \theta_F^*$ would never defect by passing in the n'th move with the plan of bidding in the next round.‖

Proof (of Lemma 3). We first examine a defection where, if FTB does make a second bid it truthfully reveals his valuation. We then show that the argument that rules out this defection also rules out all multiple bid defections.

Let the valuation of FTB be θ (F subscript omitted). Suppose that FTB contemplates a low-bid defection, signalling $\hat{\theta} < \theta$ on his first bid, with the plan to bid truthfully next round if STB responds with a counter bid. Let θ_n^* denote the lowest valuation FTB who, in equilibrium, bids at move n, $G_S(\cdot)$ the probability distribution function for STB's valuation conditional on FTB making his first bid at move n (*i.e.*, conditional on STB's valuation

[25] Given the out-of-equilibrium beliefs, FTB with $\theta_F > \theta_F^*$ optimally makes the same response to a STB that signalled $\theta_S < \theta_F^*$ as would a FTB with $\theta_F = \theta_F^*$, by bidding $\hat{\theta}_S - \gamma$.

lying in the interval $[\underline{\theta}, \theta^*_{n-1})$). In the rest of this subsection, We suppress the $n-1$ subscript on θ^* and the S subscript on probability distributions.

We now calculate the profit from the alternative strategy of defecting to a lower bid. Recall that a FTB with valuation θ^* makes an equilibrium bid of $\underline{b}$. Therefore it is impossible to signal a valuation lower than θ^* by bidding. So we consider bids that signal a valuation of $\theta^* \leq \hat{\theta} < \theta$.

If STB bids in response, we assume that FTB then either passes or counters with a further, and this time truthful signalling bid. Since bidding is costly, if FTB's valuation θ were only slightly above $\hat{\theta}$, he would not find it profitable to bid again. There is a new critical value, θ', the minimum value of θ such that FTB would be just willing to bid a second time in response to the equilibrium counter bid by STB of $\hat{\theta} - \gamma$. Let $\pi^D(\theta; \hat{\theta})$ denote FB's expected payoff from defecting to a low bid which signals a valuation of $\hat{\theta} < \theta$. Arguments almost identical to those used in Subsection 5.2 in developing equation (A12) show that

$$\pi^D(\theta; \hat{\theta}) = \pi^D(\hat{\theta}; \hat{\theta}) + G(\hat{\theta})(\theta - \hat{\theta}) + [1 - G(\hat{\theta})] \int_{\theta'}^{\theta} H(s)ds$$

where

$$H(s) = \frac{G(s) - G(\hat{\theta})}{1 - G(\hat{\theta})}$$

is the distribution of STB conditional on his valuation being greater than $\hat{\theta}$. Also, since $\pi^D(\hat{\theta}; \hat{\theta})$ is *not* a defection, this profit is equal to the equilibrium profit for FB of type $\hat{\theta}$.

If, however, FTB follows the equilibrium strategy, then by the envelope theorem, his expected profit is:

$$\begin{aligned}
\pi^E(\theta) &= \pi^E(\theta^*) + \int_{\theta^*}^{\theta} G(s)ds \\
&= \pi^E(\theta^*) + G(\hat{\theta})(\theta - \hat{\theta}) + [1 - G(\hat{\theta})] \int_{\hat{\theta}}^{\theta} H(s)ds,
\end{aligned}$$

where the E superscript denotes Equilibrium profit, and the second equality follows by simple algebra. The difference is

$$\pi^E(\theta) - \pi^D(\theta; \hat{\theta}) = [1 - G(\hat{\theta})] \int_{\theta'}^{\hat{\theta}} H(s)ds.$$

We have demonstrated that $\theta' > \hat{\theta}$, meaning this quantity must be positive. Thus, the defection to the low bid strategy is not profitable.

To verify the equilibrium, we must also show that a strategy involving *any* number of planned bids is less profitable than the equilibrium strategy. To see this, note that the second bid problem of FTB in the two-bid strategy is isomorphic to the bidding problem in the single bid strategy. So, the expected profit from the second bid will be reduced if on the second bid FTB does not reveal his true type, but instead underbids and than bids a third time if STB bids in response. This reasoning extends indefinitely to show that any strategy involving more than a single equilibrium bid will yield a lower expected profit. This completes the proof of Proposition 3. $\|$

Proof (of Proposition 4). Consider a bidder (FB or SB) with valuation θ at an arbitrary point in the game tree along the equilibrium path. In the absence of any prior bid by the other bidder, he will become FTB at some move n. For conditioning expectations, we will sometimes use an integer variable (e.g., 'n') to denote the event that the first bid occurs in the n'th move. Let $\pi(\theta|n)$ be defined as the expected profit of the given bidder if the first bid is in the n'th move, *i.e.*, he is STB. The given bidder must have parity matching n, *i.e.*, he must be FB *iff* n is odd, and SB iff n is even. For the remainder of the proof, without risk of ambiguity we suppress all subscripts on distribution and density functions. Let s denote the valuation of the other bidder. We define the conditional distribution function

$$\begin{aligned} G(\theta) &= \Pr(s < \theta|n) \\ &= \Pr(s < \theta|s < \theta^*_{n-1}) \\ &= \frac{F(\theta)}{F(\theta^*_{n-1})}. \end{aligned} \tag{A13}$$

Similarly, the density is

$$g(\theta) = \frac{f(\theta)}{F(\theta^*_{n-1})}.$$

We distinguish two cases.

Case 1 ($\theta^*_n < \theta < \theta^*_{n-1}$): In this case, the bidder makes zero profits unless he is FTB, because if the other bidder is FTB, his valuation is at least $s > \theta^*_{n-1}$. Also, in this case, if the first bid is made in the n'th move, it is never so high that the flat part of the bid schedule is reached where FTB is sure of winning. The conditional expected profit of FTB given that the first bid occurs on the n'th turn is

$$\begin{aligned} \pi(\theta|n) &= G(\theta)\left[\theta - \left(\frac{\int_{\theta^*_n}^{\theta} sg(s)ds}{G(\theta)}\right)\right] \\ &= \frac{1}{F(\theta^*_n)}\left[F(\theta)\theta - \int_{\theta^*_n}^{\theta} sf(s)ds\right]. \end{aligned} \tag{A14}$$

The unconditional expected profit is the probability-weighted sum of conditional expected profits. However, we have seen that the conditional expected profit when the bidder is not *FTB* is zero. Therefore, the expected profit is the probability of becoming *FTB*, $F(\theta^*_{n-1})$, multiplied by (A14). Thus,

$$\begin{aligned}\frac{d\pi(\theta)}{d\theta} &= F(\theta^*_{n-1})\frac{d\pi(\theta|n)}{d\theta}\\ &= F(\theta) + f(\theta)\theta - f(\theta)\theta\\ &= F(\theta).\end{aligned} \tag{A15}$$

Case 2 ($\theta^*_{n-1} < \theta < \theta^*_{n-2}$): The fact that $\theta^*_{n-1} < \theta$ leads to two key differences. First, when the first bid is made on the n'th move, the bidder is sure to win and bids on the flat part of his bid schedule. Second, the bidder may win even if the other bidder bids on move $n-1$.

Consider first the case where the other bidder has valuation $s < \theta^*_{n-1}$, which occurs with probability $F(\theta^*_{n-1})$, so that the first bid is in move n. Then *FTB*'s conditional expected profits can be written relative to the profits of a bidder with valuation θ^*_{n-1} as

$$\pi(\theta|n) = \pi(\theta^*_n|n) + \int_{\theta^*_n}^{\theta^*_{n-1}} G(s)ds + (\theta - \theta^*_{n-1}). \tag{A16}$$

Here the first term is the conditional expected profit of a bidder with valuation θ^*_n. By a standard envelope condition argument, since we are examining a revealing continuous bid schedule, a bidder with valuation θ^*_{n-1} will have profits equal to this first profit plus the second term above (the integral). This integrates just up to the point where the conditional distribution function hits 1. Finally, since the bidder actually has valuation $\theta > \theta^*_{n-1}$, he makes an additional profit given by the last term. He makes this additional profit with certainty since he is sure to win. (The last quantity could be folded into the integral by making the upper limit of integration be θ.)

Consider next the case where the other bidder has valuation $\theta^*_{n-1} < s < \theta^*_{n-3}$, which occurs with probability $F(\theta^*_{n-3}) - F(\theta^*_{n-1})$, so that the first bid is in move $n-1$. (If s is above this range, the other bidder will win for sure, and the profits we are calculating are zero.)

Now define the conditional distribution function $H(\theta)$ as the probability the bidder is high given that the other bidder bid first in move $n-1$, *i.e.*,

$$\begin{aligned}H(\theta) &= \Pr(s < \theta|n-1)\\ &= \Pr(s < \theta|\theta^*_{n-1} < s < \theta^*_{n-3})\\ &= \frac{F(\theta)}{F(\theta^*_{n-3}) - F(\theta^*_{n-1})}.\end{aligned}$$

Similarly, the density

$$h(\theta) = \frac{f(\theta)}{F(\theta^*_{n-3}) - F(\theta^*_{n-1})}.$$

Now the bidder's conditional expected profit can be calculated by integrating over possible values of FTB at $n-1$,

$$\begin{aligned}\pi(\theta|n-1) &= \int_{\theta^*_{n-1}}^{\theta} (\theta - s)h(s)ds \\ &= \theta[H(\theta) - H(\theta^*_{n-1})] - \int_{\theta^*_{n-1}}^{\theta} sh(s)ds \\ &= \frac{1}{F(\theta^*_{n-3}) - F(\theta^*_{n-1})}\left(\theta[F(\theta) - F(\theta^*_{n-1})] - \int_{\theta^*_{n-1}}^{\theta} sf(s)ds\right) \quad \text{(A17)}\end{aligned}$$

Since unconditional expected profit is a probability weighted average of conditional expected profits, we multiplying (A16) by $F(\theta^*_{n-1})$, multiply (A17) by $F(\theta^*_{n-3}) - F(\theta^*_{n-1})$, sum, and integrate by parts the last integral in (A17). This gives

$$\begin{aligned}\pi(\theta) &= \pi(\theta^*_n|n)F(\theta^*_{n-1}) + \int_{\theta^*_n}^{\theta^*_{n-1}} F(s)ds + \theta F(\theta^*_{n-1}) - \theta^*_{n-1}F(\theta^*_{n-1}) + \theta F(\theta) - \theta F(\theta^*_{n-1}) \\ &\quad - \int_{\theta^*_{n-1}}^{\theta} sf(s)ds \\ &= \pi(\theta^*_n|n)F(\theta^*_{n-1}) + \int_{\theta^*_n}^{\theta} F(s)ds \\ &= \pi(\theta^*_n) + \int_{\theta^*_n}^{\theta} F(s)ds,\end{aligned}$$

where the last equality holds because

$$\pi(\theta^*_n) = \pi(\theta^*_n|n)F(\theta^*_{n-1}) + \pi(\theta^*_n|s > \theta^*_{n-1})[1 - F(\theta^*_{n-1})], \quad \text{(A18)}$$

and the last term on the RHS is zero, because if $s > \theta^*_{n-1}$, then the bidder with $\theta = \theta^*_n$ always loses so his conditional profit in the above equation is zero.

Differentiating with respect to θ shows that just as in Case 1, equation (A15),

$$\frac{d\pi(\theta)}{d\theta} = F(\theta). \quad \text{(A19)}$$

Since equation (A19) obtains for values of θ satisfying either Case 1 or Case 2, and n is arbitrary, and since $\lim_{n\to\infty} \theta^*_n = \max\{\underline{\theta}, \underline{b} + \gamma\}$, (A19) holds for all $\theta > \underline{b} + \gamma$. Thus,

$$\pi(\theta) = \int_{\underline{b}+\gamma}^{\theta} F(\theta) d\theta.$$

This last quantity is the profit to a bidder in a FPSB auction with minimum bid of $\underline{b} + \gamma$.

Thus, the auctions are bidder-profit equivalent. Since stochastic deadweight costs are incurred, the CSB auction yields lower expected revenues for the seller by the expected bid costs incurred. $\|$

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