Liquidity regimes and optimal dynamic asset allocation☆

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Abstract

We solve a portfolio choice problem when expected returns, covariances, and trading costs follow a regime-switching model. The optimal policy trades towards an aim portfolio given by a weighted-average of the conditional mean-variance portfolios in all future states. The trading speed is higher in more persistent, riskier, and higher-liquidity states. It can be optimal to overweight low Sharpe-ratio assets such as Treasury bonds because they remain liquid even in crisis states. We illustrate our methodology by constructing an optimal US equity market timing portfolio based on an estimated regime-switching model and on trading costs estimated using a large-order institutional trading data set.

Keywords: Portfolio choice, Dynamic models, Transaction costs, Stochastic volatility, Price impact, Risk-parity, Mean-variance.

JEL classification: D53, G11, G12.

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1. Introduction

Mean-variance efficient portfolio optimization, introduced by Markowitz (1952), is still widely used in practice and taught in business schools. When either expected returns or the covariance matrix of returns changes over time then so will the conditional mean-variance efficient 'Markowitz' portfolio. However, when trading costs are non-zero, it is not optimal to rebalance rebalance so as to perfectly track the Markowitz portfolio. In recognition of this fact, practitioners generally employ ad hoc adjustments to Markowitz optimization, but it is recognized that these approaches are not optimal (Grinold and Kahn, 1999).

In a recent paper, Gârleanu and Pedersen (2013, GP) show that in the presence of quadratic transaction costs,[1] an investor with mean-variance preferences should adopt a trading rule that only partially rebalances from her current position towards an *aim portfolio* at a fixed *trading speed*.[2] They derive closed-form expressions for both the optimal aim portfolio and the trading speed that depend on the dynamics of expected returns, the quantity of and aversion to risk, and the magnitude of price impact. However, the GP model assumes that both the covariance matrix of price changes and the price-impact parameters are constant. In this paper we derive a closed-form solution for the optimal portfolio trading rule in a similar setting but where, in addition to expected returns, volatilities and transaction costs may be stochastic. This is consistent with considerable empirical evidence that stock return volatilities are stochastic and that transaction costs covary with the level of stock volatility, going back at least to Rosenberg (1972) for the former and to Stoll (1978) for the latter.

Specifically, we obtain a closed-form solution for the optimal dynamic portfolio when expected returns, covariances, and price impact parameters follow a multi-state Markov switching model.[3] Consistent with GP, we assume that the investor's objective function is 'dynamic' mean-variance: investors maximize the expected discounted sum of portfolio returns net of trading costs, minus a penalty for the variance of portfolio returns.[4]

In this setting, and for an agent with these preferences, we show that the optimal trading rule is similar to that derived in GP, namely to partially trade from the current position towards an aim portfolio. However, when volatilities and trading costs are stochastic, then both the aim portfolio and the trading speed are state dependent. Specifically, the aim portfolio is a weighted average of the state-contingent Markowitz portfolios in all possible future states, where the weight on each conditional-Markowitz portfolio is a function of the likelihood of transitioning to that state, the state persistence, and the risk and transaction costs faced in that state relative to the current one. Similarly, the optimal trading speed depends on the relative magnitude of the transaction costs in various states and their transition probabilities. Moreover, while we solve the model

[1] Quadratic transaction costs emerge with a linear price impact model, i.e., trading Δ shares of a stock move its average price by $\lambda\Delta$ for a given constant λ.

[2] Litterman (2005) makes a similar point in an unpublished note.

[3] One additional advantage of this approach, which we discuss more below, and develop fully in Section 4, is that it allows us to specify a process for *return* volatility as opposed to *price-change* volatility in each state, consistent with the observed log-normality of prices.

[4] In the Internet Appendix, we discuss two approaches to micro-found this objective function, which is frequently used by academics and practitioners.

in a discrete-time setting in the body of the paper, Internet Appendices B.1 and B.2 solve continuous time versions of the model, and obtain consistent solutions.

To see how the solution here differs from the GP solution, which assumes constant volatility and liquidity, consider a simple setting with a single risky asset and with two states: a low volatility state L where transaction costs are zero, and a high volatility state H where transaction costs are positive. When the economy is in the L-state, it is clearly optimal to trade (at infinite speed) all the way to the aim portfolio because transaction costs in that state are zero. In contrast, trading speed will be finite in the H-state. Further, the aim portfolio in the H-state will equal the conditional Markowitz portfolio in that state.[5] Intuitively, in the H-state the investor should put zero-weight on the L-state Markowitz portfolio, because when the economy does enter the L-state she can immediately rebalance to the (optimal) aim portfolio at zero cost. However, the aim portfolio in the L-state will be a weighted average of both H- and L-conditional Markowitz portfolios, where the weight on the H-conditional Markowitz portfolio increases with the likelihood of transitioning from L to H, the persistence of the state H, and with the ratio of the volatilities in the H- and L-states.

One immediate implication of our model is that the aim portfolio will deviate significantly from the Markowitz benchmark in anticipation of possible future shifts in relative risk and/or transaction costs. Consider two assets, which can be thought of as "Treasury" and "Corporate" bond portfolios. Suppose that in the low-volatility state (state L), the Corporate portfolio has slightly lower liquidity, but a far higher Sharpe ratio than the Treasury portfolio, so that the conditional Markowitz portfolio has most of its weight on Corporates. However, if the economy transitions to state H, then the risk and trading costs will rise and the Sharpe-ratio will both rise for Corporates, but all will remain unchanged for Treasuries. We first show that, in anticipation of this, the aim portfolio in the L-state will have a large Treasury position. Intuitively, if the economy transitions from the L- to the H-state, then the volatility of the Corporate portfolio will increase, its Sharpe ratio will fall, and it will become illiquid and costly to trade out of. Thus, the aim portfolio preemptively reduces the holdings of Corporates in the L-state.

Second, while GP show that in their setting trading speed is a function only of the (constant) trading cost and volatility, in our setting trading speed takes account of both current and future values of these parameters. Continuing this example, in the L-state it optimal to trade the less liquid Corporate portfolio more aggressively than the Treasury portfolio because, if the economy does transition to the H-state, the Corporate portfolio will become much more expensive to trade, while the Treasury portfolio will remain relatively liquid.

Our model also has implications for the popular (among practitioners) "risk-parity" strategy, which weights each asset class in such a way that each contributes an equal amount of volatility to the overall fund (see, e.g., Bridgewater, 2011; Asness, Frazzini and Pedersen, 2012). Risk-parity can be thought of as the mean-variance efficient portfolio, when all asset classes have identical Sharpe ratios and the correlations across asset classes

[5] That is, the aim portfolio in the H-state puts zero weight on the L-state Markowitz portfolio.

are zero.[6] Interestingly, even if it were optimal to hold a risk-parity portfolio at all times in the absence of transaction costs, we show that, when transaction costs and volatilities of various asset classes move over time in a correlated fashion, then it is optimal to deviate significantly from the risk-parity portfolio, and that this deviation is larger in the low-risk regime. This is because the optimal portfolio in the low-risk regime, where transaction costs tend to be lowest, needs to put some weight on the optimal risk-parity portfolio in the high-risk regime, where high transaction costs will make it much more costly to delever out of the higher risk asset classes.

We present an empirical application of our framework in which a fund moves in and out of a stock market index, taking into account time-varying expected returns, volatility, and transaction costs. While our analytical results are all derived in the context of a regime-switching model of price changes (i.e., a Gaussian normal model for prices), we show that our model remains tractable for a regime-switching model of dollar returns (a log-normal model of prices). Since the latter model fits the data empirically better, we use this framework for the empirical implementation. We estimate a four-state Markov regime-switching model of returns and find, both in-sample and out-of-sample, evidence of time-variation in first and second moments. To estimate the transaction cost parameters, we use a proprietary data set on realized trading costs incurred by a large financial institution trading on behalf of clients, as measured by the implementation shortfall of their trades (Perold, 1988). We show that trading costs vary significantly across regimes and that, not surprisingly, trading costs are higher for higher volatility regimes.

We test our trading strategy both in-sample and out-of-sample. For the out-of-sample test, the regime shifting model and the state probabilities are estimated using only data in the information set of an agent on the day preceding the trading date. We compare the performance of our optimal dynamic strategy to three alternatives: a constant-dollar investment in the risky asset, corresponding to an unconditional estimate of the sample mean and variance of returns; a buy-and-hold policy that never trades; and finally a myopic one-period mean-variance problem optimized for current transaction costs, but that ignores the future dynamics of the Markov regime-switching model (see, e.g., Grinold and Kahn, 1999).

We find that the net-of-cost performance of the dynamic trading strategy is far higher than the other three strategies. To determine the source of this superior performance, we examine what source of time-variation leads to the biggest gains for the dynamic strategy. Specifically, we compare the gains obtained from timing changes in expected returns, in volatility, and in transaction costs. In this out-of-sample experiment, we find that the biggest benefits arise from taking into account for time-variation in market volatility and transaction costs, while the benefits from timing (estimated) variation in mean returns is more mixed. This reflects the fact that mean returns move less than one-for-one with variances. Our findings here are consistent with Moreira and Muir (2017), who show that there are gains to moving out of the market in response to an increase in market variance because the conditional market risk-premium moves less than one-for-one with its variance.

[6] These assumptions are sometimes justified based on the difficulty to reliably estimate means and correlations.

Thus, since our model captures the time-variation in volatilities and the corresponding changes in transaction costs more accurately, it is able to manage the risk-exposure and the incurred transaction costs more reliably, which directly contributes to increasing the net performance.

There is a large academic literature on portfolio choice that has extended Markowitz's one-period mean-variance setting to a dynamic multiperiod setting with a time-varying investment opportunity set and more general objective functions.[7] This literature has largely ignored realistic frictions such as trading costs, because introducing transaction costs and price impact in the standard dynamic portfolio choice problem tends to make it intractable. Indeed, most academic papers studying transaction costs focus on a very small number of assets (typically two), limited predictability, and typically no time-variation in second moments or transaction costs.[8]

Balduzzi and Lynch (1999) and Lynch and Balduzzi (2000) investigate the impact of fixed and proportional transaction costs on the utility costs and the optimal rebalancing rule of a single risky asset with time-varying expected return, using dynamic programming. Lynch and Tan (2010) use a numerical procedure to solve for the optimal portfolio choice of an investor with access to two risky assets under return predictability and proportional transaction costs. Brown and Smith (2011) discuss the high-dimensionality of the problem and provide heuristic trading strategies and dual bounds for a general dynamic portfolio optimization problem with transaction costs and return predictability that can be applied to a larger number of stocks. Longstaff (2001) studies a numerical solution to the one risky asset case with stochastic volatility when agents face liquidity constraints that force them to trade absolutely continuously.

Our paper is also related to the large literature on asset allocation under regime shifts. For example, Ang and Bekaert (2002) apply a regime-switching model to an international asset allocation problem to account for time-varying first and second moments of asset returns. Ang and Timmermann (2012) survey this literature in detail. One common observation in empirical work estimating regimes is the low expected returns in high-volatility states. Thus, these models would often suggest that the mean-variance investors should scale down their equity exposure in times of market stress. Our paper complements this literature by accounting for high transaction costs during these volatile periods. Jang, Keun Koo, Liu and Loewenstein (2007) extend the models of Constantinides (1986) and Davis and Norman (1990) (e.g., one risky asset and one risk-free asset) with regime-switching fundamental parameters. They consider a small investor with no price impact and illustrate that proportional transaction costs may have a first-order effect on liquidity premia. In comparison, we consider a regime-switching model in which an investor with price impact can trade multiple risky assets.

As noted earlier, our paper is most closely related to Litterman (2005) and Gârleanu and Pedersen (2013,

[7] Merton (1969, 1971), Brennan, Schwartz and Lagnado (1997), Kim and Omberg (1996), Campbell and Viceira (2002), Campbell, Chan and Viceira (2003), Liu (2007), Detemple and Rindisbacher (2010), and many more. See Cochrane (2007) for a survey.

[8] Constantinides (1986), Davis and Norman (1990), Dumas and Luciano (1991), and Shreve and Soner (1994) study the two-asset (one risky, one risk-free) case with independent and identically distributed (i.i.d.) returns. Liu (2007) studies the multi-asset case under constant absolute risk aversion (CARA) preferences and i.i.d. returns. Cvitanić (2001) surveys this literature.

GP). They obtain a closed-form solution for the optimal portfolio choice in a model where: (1) expected price change per share for each security is a linear, time-invariant function of a set of autoregressive predictor variables; (2) the covariance matrix of price changes is constant; (3) trading costs are a time-invariant quadratic function of the number of shares traded, and (4) investors have a linear-quadratic objective function. Their approach relies heavily on linear-quadratic stochastic programming (see, e.g., Ljungqvist and Sargent, 2004). Our approach uses a similar objective function, but allows for time-variation in means, volatilities, and transaction costs, albeit within a regime-switching framework. Moreover, in contrast with the GP framework, our framework is equally tractable when expected price changes are constant in each state of the regime-switching model (i.e., prices follow arithmetic Brownian motion) or when expected returns, conditional on the state, are constant (i.e., prices follow geometric Brownian motion). Because historical returns are better described with a log-normal distribution, this prices is Since the latter is a more realistic description of historical returns, it is the one we use for our empirical implementation.

2. A regime switching model for price changes

We begin with a setting with N risky assets, in which the N-dimensional vector of price changes from period t to $t+1$, dS_t, follows the process:

$$\begin{aligned} \mathrm{E}[dS_t] &= \mu(s_t) \\ \mathrm{E}[(dS_t - \mu(s_t))(dS_t - \mu(s_t))^\top] &= \Sigma(s_t), \end{aligned}$$

where $\mu(s_t)$ and $\Sigma(s_t)$ are, respectively, the N-vector of expected price changes and the $N \times N$ covariance matrix of price changes. Both μ and Σ are a function of a state variable s_t which follows a Markov chain with transition probabilities $\pi_{s,s'}$. In Section 4, we will solve for the optimal dynamic strategy when returns, rather than price change, follow this process.

We consider the optimization problem of an agent with the following objective function:

$$\max_{n_t} \mathrm{E}\left[\sum_{t=0}^{\infty} \rho^t \left\{ n_t^\top \mu(s_t) - \frac{1}{2}\gamma n_t^\top \Sigma(s_t) n_t - \frac{1}{2}\Delta n_t^\top \Lambda(s_t) \Delta n_t \right\}\right]. \tag{1}$$

This objective function is the same as that considered by GP, namely, that of an investor who maximizes a discounted sum of mean-variance criterion in every period, net of trading costs. It is also popular among practitioners (e.g., Litterman, 2005). In the Internet Appendix, we show that in the continuous time limit of the model this objective function corresponds to an agent who maximizes her expected wealth $\mathrm{E}[W_\tau]$ at some random horizon τ, drawn from an exponential distribution with intensity $-\ln\rho > 0$, who faces quadratic transaction costs and incurs continuous holding costs that are proportional to the variance of the position.[9]

[9] These holding costs are justified by the fact that even though asset management firms may not have direct risk-aversion,

The agent chooses her holdings n_t in each period t so as to maximize the objective function in Eq. (1). Specifically, at the end of period $t-1$, the agent hold n_{t-1} shares of the N assets. At this point the agent observes the state s_t, and trades $\Delta n_t = n_t - n_{t-1}$ shares. As noted earlier, consistent with GP we specify a linear price impact model. $\Lambda(s_t)$ is the price impact matrix, so the N-vector of price concessions is $\Lambda(s_t)\Delta n_t$ and the total (dollar) cost of trading in period t is therefore $\frac{1}{2}\Delta n_t^\top \Lambda(s_t)\Delta n_t$. We assume that Σ_s and Λ_s are real symmetric positive-definite matrices.[10]

In the absence of transaction costs (when $\Lambda_s = 0$), the optimal solution would be to hold the conditionally mean-variance optimal Markowitz portfolio $m_s = (\gamma\Sigma_s)^{-1}\mu_s$ at all times. Further, if there were no time-variation in the investment opportunity set (that is, if μ_s and Σ_s were constant), then it would be always optimal to hold the mean-variance efficient Markowitz portfolio. However, when there are transaction costs and the opportunity set is time-varying, it becomes optimal for the investor to rebalance the portfolio, and deviate from the conditionally mean-variance efficient portfolio.

In the GP framework, the conditional mean of stock price changes (μ_s) follows an AR(1) process, but the covariance matrix Σ and the matrix of transaction cost parameters Λ are required to be deterministic. In our framework Σ and Λ vary across states. Using a Markov regime-switching model allows us to obtain tractable solutions even though the model is not in the standard linear-quadratic framework.

For simplicity we begin by considering only a two-state Markov chain model, with states H and L, but we generalize this to more states in Section 2.4. We will use the following notation throughout: for all t where $s_t = s \in \{H, L\}$, $s_{t+1} = z \in \{H, L\}$ and $s' = \{H, L\} \setminus s$. Then, using the dynamic programming principle, the value function $V(n_{t-1}, s)$ satisfies

$$V(n_{t-1}, s) = \underset{n_t}{\text{maximize}} \left(n_t^\top \mu_s - \frac{1}{2}\Delta n_t^\top \Lambda_s \Delta n_t - \frac{\gamma}{2} n_t^\top \Sigma_s n_t + \rho \mathbb{E}\left[V_t(n_t, z)\right] \right).$$

We guess the following quadratic form for our value functions:

$$V(n, s) = -\frac{1}{2} n^\top Q_s n + n^\top q_s + c_s,$$

where Q_s is a symmetric $N \times N$ matrix and q_s, c_s are N-dimensional vectors of constants for $s \in \{H, L\}$. We now define the expectation conditional on state s for any matrix M_s to be $\overline{M}_s = \pi_{s,s} M_s + \pi_{s,s'} M_{s'}$. With this notation, the right-hand side of the Hamilton-Jacobi-Bellman (HJB) equation we are optimizing can be

they face extra costs for holding inventory in risky assets, that may be related to regulatory capital requirements, prime broker fees, collateral and financing costs, or risk-budgeting constraints. Quadratic holding costs have been widely used in the literature (e.g., Vives, 2011; Rostek and Weretka, 2012; Du and Zhu, 2017; Duffie and Zhu, 2017). In our formulation holding costs are effectively also quadratic, but unlike in these other papers, they are stochastic because they change with the liquidity and risk state. As an alternative micro-foundation for this objective function based on risk preferences, we show in the Internet Appendix, that it corresponds to maximizing the certainty equivalent of an agent with 'source-dependent' stochastic differential utility who has a constant absolute risk-aversion coefficient γ towards return shocks and vanishing risk-aversion coefficient $\gamma_2 \to 0$ towards regime-shifts.

[10] Naturally, we want $\theta^\top \Lambda \theta > 0 \ \forall\ \theta \neq 0$. Further, we have $\theta^\top \Lambda \theta = \frac{1}{2}\theta^\top \Lambda \theta + \frac{1}{2}(\theta^\top \Lambda \theta)^\top = \theta^\top (\frac{1}{2}\Lambda + \frac{1}{2}\Lambda^\top)\theta$. So if Λ is not symmetric we can replace it with $\frac{1}{2}(\Lambda + \Lambda^\top)$ which is.

rewritten as a quadratic objective:

$$-\frac{1}{2}n_t^\top J_s n_t + n_t^\top j_s + k_s$$

where

$$\begin{aligned}
J_s &= \gamma\Sigma_s + \Lambda_s + \rho\overline{Q}_s \\
j_s &= \mu_s + \Lambda_s n_{t-1} + \rho\overline{q}_s \\
k_s &= -\frac{1}{2}n_{t-1}\Lambda_s n_{t-1} + \rho\overline{c}_s.
\end{aligned}$$

This is optimized for $n_t = J_s^{-1}j_s$, that is:

$$n_t = \left(\gamma\Sigma_s + \Lambda_s + \rho\overline{Q}_s\right)^{-1}\left(\mu_s + \rho\overline{q}_s + \Lambda_s n_{t-1}\right).$$

Further, the optimized value is simply $\frac{1}{2}j_s^\top J_s^{-1}j_s + k_s$. Thus matching coefficients we find that the matrices Q_s, q_s for $s = H, L$ must satisfy the system of equations:

$$Q_s = -\Lambda_s\left(\gamma\Sigma_s + \Lambda_s + \rho\overline{Q}_s\right)^{-1}\Lambda_s + \Lambda_s, \tag{2}$$

$$q_s = \Lambda_s\left(\gamma\Sigma_s + \Lambda_s + \rho\overline{Q}_s\right)^{-1}\left(\mu_s + \rho\overline{q}_s\right). \tag{3}$$

Note that given a solution for Q_H and Q_L, we can obtain q_H and q_L in closed-form as a matrix weighted average of μ_H and μ_L. While we are not aware of a closed-form solution for Q_H and Q_L in general, it is straightforward to obtain a numerical solution to the coupled Riccatti matrix equation, as we discuss in Lemma 2 below. Further, for a variety of special cases we consider below, it is possible to obtain closed-form solutions.

With a solution in hand, we can define the conditional aim portfolio as the portfolio that maximizes the value function at any time t conditional on the state. We can now characterize the optimal trading rule and the aim portfolios.

Theorem 1. *The optimal trade at time t in state s is a matrix weighted average of the current position vector and the conditional aim portfolio:*

$$n_t = (I - \tau_s)n_{t-1} + \tau_s aim_s \tag{4}$$

where the trading speed $\tau_s = I$ (and $Q_s = 0$) if $\Lambda_s = 0$, and else $\tau_s = \Lambda_s^{-1}Q_s$ $\forall s = \{H, L\}$ where (Q_H, Q_L) solve a system of coupled equations:

$$I - \Lambda_s^{-1}Q_s = [\Lambda_s^{-1}(\gamma\Sigma_s + \rho\pi_{ss'}Q_{s'}) + I + \rho\pi_{ss}\Lambda_s^{-1}Q_s]^{-1}. \tag{5}$$

The aim portfolio, which maximizes the value function conditional on the current state, is given by

$$aim_s = \left(\gamma\Sigma_s + \rho\overline{Q}_s\right)^{-1}(\mu_s + \rho\overline{q}_s). \tag{6}$$

Further, the aim portfolio is a weighted average of the conditional Markowitz portfolios ($m_s = (\gamma\Sigma_s)^{-1}\mu_s$):

$$aim_s = (I - \alpha_s)m_s + \alpha_s m_{s'} \ \ \forall s = H, L \tag{7}$$

where

$$\alpha_s = \{(\gamma + \rho\pi_{s's}Q_{s'}\Sigma_{s'}^{-1}Q_sQ_{s'}^{-1})\Sigma_s + \rho\pi_{ss'}Q_{s'}\}^{-1}\rho\pi_{ss'}Q_{s'}$$

.

Proof. Optimizing the value function with respect to n_t gives:

$$aim_s = (Q_s)^{-1}(q_s) \ \ \forall s = H, L.$$

Substituting from the definitions in Eqs. (2) and (3) we obtain:

$$\begin{aligned} aim_s &= \left(-\Lambda_s\left(\gamma\Sigma_s + \Lambda_s + \rho\overline{Q}_s\right)^{-1}\Lambda_s + \Lambda_s\right)^{-1}\left(\Lambda_s\left(\gamma\Sigma_s + \Lambda_s + \rho\overline{Q}_s\right)^{-1}(\mu_s + \rho\overline{q}_s)\right) \\ &= \left(-\left(\gamma\Sigma_s + \Lambda_s + \rho\overline{Q}_s\right)^{-1}\Lambda_s + I\right)^{-1}\left(\gamma\Sigma_s + \Lambda_s + \rho\overline{Q}_s\right)^{-1}(\mu_s + \rho\overline{q}_s) \\ &= \left(\gamma\Sigma_s + \rho\overline{Q}_s\right)^{-1}(\mu_s + \rho\overline{q}_s) \end{aligned}$$

where the last equality obtains by noting that if we define the matrix

$$M = \left(-\left(\gamma\Sigma_s + \Lambda_s + \rho\overline{Q}_s\right)^{-1}\Lambda_s + I\right)^{-1}\left(\gamma\Sigma_s + \Lambda_s + \rho\overline{Q}_s\right)^{-1}$$

then

$$M^{-1} = \left(\gamma\Sigma_s + \Lambda_s + \rho\overline{Q}_s\right)\left(-\left(\gamma\Sigma_s + \Lambda_s + \rho\overline{Q}_s\right)^{-1}\Lambda_s + I\right) = \left(\gamma\Sigma_s + \rho\overline{Q}_s\right),$$

which immediately implies that $M = \left(\gamma\Sigma_s + \rho\overline{Q}_s\right)^{-1}$.

We then expand the expression for aim_s:

$$\begin{aligned} & aim_s = \left(\gamma\Sigma_s + \rho\pi_{ss}Q_s + \rho\pi_{ss'}Q_{s'}\right)^{-1}\left(\mu_s + \rho\overline{Q}_s\right) \\ \Rightarrow \quad & \left(\gamma\Sigma_s + \rho\pi_{ss}Q_s + \rho\pi_{ss'}Q_{s'}\right)aim_s = \left(\gamma\Sigma_s m_s + \rho\pi_{ss}Q_s aim_s + \rho\pi_{ss'}Q_{s'}aim_{s'}\right) \\ \Rightarrow \quad & \left(\gamma\Sigma_s + \rho\pi_{ss'}Q_{s'}\right)aim_s = \left(\gamma\Sigma_s m_s + \rho\pi_{ss'}Q_{s'}aim_{s'}\right) \\ \Rightarrow \quad & aim_s = \left(\gamma\Sigma_s + \rho\pi_{ss'}Q_{s'}\right)^{-1}\left(\gamma\Sigma_s m_s + \rho\pi_{ss'}Q_{s'}aim_{s'}\right). \end{aligned}$$

We then substitute for $aim_{s'} = \left(\gamma\Sigma_{s'} + \rho\pi_{s's}Q_s\right)^{-1}\left(\gamma\Sigma_{s'}m_{s'} + \rho\pi_{s's}Q_s aim_s\right)$ and obtain after dividing by

γ

$$\left[\Sigma_s + \frac{\rho}{\gamma}\pi_{ss'}Q_{s'}\left(I - \left(\gamma\Sigma_{s'} + \rho\pi_{s's}Q_s\right)^{-1}\rho\pi_{s's}Q_s\right)\right] aim_s = \Sigma_s m_s + \rho\pi_{ss'}Q_{s'}\left(\gamma\Sigma_{s'} + \rho\pi_{s's}Q_s\right)^{-1}\Sigma_{s'}m_{s'}.$$

Using the simple identity $I - (F+G)^{-1}G = (F+G)^{-1}F$, with $F = \gamma\Sigma_{s'}$ and $G = \rho\pi_{s's}Q_s$, we finally obtain

$$\left\{\Sigma_s + \rho\pi_{ss'}Q_{s'}[\gamma\Sigma_{s'} + \rho\pi_{s's}Q_s]^{-1}\Sigma_{s'}\right\} aim_s = \Sigma_s m_s + \rho\pi_{ss'}Q_{s'}[\gamma\Sigma_{s'} + \rho\pi_{s's}Q_s]^{-1}\Sigma_{s'}m_{s'}.$$

Thus, this shows that we can write $aim_s = (I - \alpha_s)m_s + \alpha_s m_{s'}$ where

$$\alpha_s = \left\{\Sigma_s + \rho\pi_{ss'}Q_{s'}[\gamma\Sigma_{s'} + \rho\pi_{s's}Q_s]^{-1}\Sigma_{s'}\right\}^{-1}\rho\pi_{ss'}Q_{s'}[\gamma\Sigma_{s'} + \rho\pi_{s's}Q_s]^{-1}\Sigma_{s'}$$

which can be further simplified to

$$\alpha_s = \{(\gamma + \rho\pi_{s's}Q_{s'}\Sigma_{s'}^{-1}Q_sQ_{s'}^{-1})\Sigma_s + \rho\pi_{ss'}Q_{s'}\}^{-1}\rho\pi_{ss'}Q_{s'}. \qquad \square$$

Eq. (4) shows that this optimal dynamic strategy is to trade to a portfolio with shares n_t that is a linear combination of the current portfolio n_{t-1} and of the aim portfolio aim_s. τ_s is the matrix that specifies how quickly the investor should trade towards the aim portfolio. $\tau_s = I$ means that, in state s, the investor should immediately and fully trade to aim_s. $\tau_s = 0$ means that the investor should not trade.

The state-contingent aim portfolio aim_s is defined as the portfolio that would maximize the value function in that state. Another interpretation of the aim portfolio is as the *no-trade* portfolio, i.e., the portfolio for which the optimal trade is zero, as long as the state does not change.[11] The speed at which we trade towards the aim portfolio is, in general, dependent on the state. That is, it is typically increasing in variance and decreasing in the transaction costs, which may be state-dependent in our framework. In the case (similar to GP) where only expected returns are stochastic (and covariances and transaction costs are constant) the trading speed is constant as well. Further, the aim portfolio is state-dependent. When either a state is absorbing ($\pi_{ss} = 1$) or transaction costs are zero ($\Lambda_s = 0$) then the aim portfolio is equal to the conditional mean-variance Markowitz portfolio (m_s). But in general, the aim portfolio is a weighted average of the conditional mean-variance portfolio across states, where the weight on each state is typically higher, if the variance of returns or the transaction cost is higher in that state.

We now consider a few special cases to gain further insights into the optimal trading rule.

[11] Note that, because the vector of security holdings n has units of shares, and because the price change process is a function only of the state, the optimal portfolio will not change when prices change.

2.1. The case where only μ_s changes with the state (GP)

If only μ_s changes with the state (i.e., if $\Sigma_s = \Sigma$ and $\Lambda_s = \Lambda$ for all s) then the solution $Q_s = Q$ is independent of the state and satisfies:

$$I - \Lambda^{-1}Q = [\gamma\Lambda^{-1}\Sigma + I + \rho\Lambda^{-1}Q]^{-1}.$$

This equation has an explicit solution as we show in the following lemma.[12]

Lemma 1. *Consider the diagonalization of the matrix $\Lambda^{-1}\Sigma = F\,\mathrm{diag}(\ell_i)\,F^{-1}$ in terms of its eigenvalues ℓ_i $\forall i = 1,\ldots,n$. Then note that*

$$I - F^{-1}\Lambda^{-1}QF = [\gamma\,\mathrm{diag}(\ell_i) + I + \rho F^{-1}\Lambda^{-1}QF]^{-1}.$$

It follows that $Q = \Lambda F\,\mathrm{diag}(\eta_i)F^{-1}$ such that the η_i solve the quadratic equations ($\forall i = 1,\ldots,n$):

$$1 - \eta_i = [\gamma\ell_i + 1 + \rho\eta_i]^{-1}$$

that is:

$$\eta_i = \frac{\rho - 1 - \ell_i\gamma + \sqrt{(\rho - 1 - \ell_i\gamma)^2 + 4\ell_i\gamma\rho}}{2\rho}.$$

This implies that the trading speed $\tau_s = \Lambda_s^{-1}Q_s = F\,\mathrm{diag}(\eta_i)F^{-1}$ is independent of the state. That is, investors trade at a constant speed towards their aim portfolio independent of the state. The speed of trading for specific stock i is increasing in the agent's time discount rate and in the agent's risk-aversion. Furthermore, for the special case where Λ and Σ are diagonal matrices, then the speed of trading stock i is increasing in $\ell_i = \Sigma_{ii}/\Lambda_{ii}$, that is, the ratio of a stock's variance to its cost of trading.

While the trading speed is constant, the aim portfolios differ across states. Indeed, using Theorem 1, the aim portfolio in state s can be computed as:

$$aim_s = (I - \alpha_s)m_s + \alpha_s m_{s'}$$

[12] We note that since Σ, Λ are assumed to be symmetric matrices with (strictly) positive real eigenvalues, then $\Lambda^{-1}\Sigma$ is diagonalizable. First, note that since Λ is real symmetric positive definite then so is its inverse. This implies we can decompose $\Lambda^{-1} = M^{\frac{1}{2}}M^{\frac{1}{2}}$. It follows that $M^{\frac{1}{2}}\Sigma M^{\frac{1}{2}}$ is symmetric and positive definite (as $x^\top M^{\frac{1}{2}}\Sigma M^{\frac{1}{2}}x = (M^{\frac{1}{2}}x)^\top\Sigma(M^{\frac{1}{2}}x) > 0\,\forall x \neq 0$ since Σ is positive definite) and therefore has positive real eigenvalues. In turn, it is easy to show that $\Lambda^{-1}\Sigma = M^{\frac{1}{2}}M^{\frac{1}{2}}\Sigma$ has the same eigenvalues as $M^{\frac{1}{2}}\Sigma M^{\frac{1}{2}}$.

where

$$\begin{aligned}\alpha_s &= \{\gamma\Sigma + \rho\pi_{s's}Q + \rho\pi_{ss'}Q\}^{-1}\rho\pi_{ss'}Q \\ &= (\gamma Q^{-1}\Sigma + (\rho\pi_{s's} + \rho\pi_{ss'})I)^{-1}\rho\pi_{ss'} \\ &= F \operatorname{diag}\left(\frac{\rho\pi_{ss'}}{\gamma\ell_i/\eta_i + \rho\pi_{s's} + \rho\pi_{ss'}}\right) F^{-1}.\end{aligned}$$

The state s aim portfolio is a weighted average of the conditional Markowitz portfolios in the current state (s) and in the alternative state (s'), where the weight on the current state Markowitz portfolio is increasing in the persistence of that state $\pi_{s,s}$ and in risk-aversion γ, but decreasing in the time discount factor ρ, and the persistence of the other state $\pi_{s',s'}$. Furthermore, the weight is also stock-specific and increasing for stock i in ℓ_i, which captures the notion that the more risky a stock is relative to its trading cost the more weight we should put on the conditional Markowitz portfolio for computing the aim portfolio.

To a large extent these results are consistent with the findings of GP, albeit with a different model of the time-variation in expected returns. The more interesting case is when we also allow covariances and transaction costs to change across states. In that case, both trading speed and aim portfolios change across states.

2.2. The case where $\Lambda_L = 0$ and $\Lambda_H > 0$

When transaction costs are zero in state L, then the solution implies $Q_L = 0$ and that Q_H solves a one-dimensional equation:

$$I - \Lambda_H^{-1}Q_H = [\gamma\Lambda_H^{-1}\Sigma_H + I + \rho\pi_{HL}\Lambda^{-1}Q_H]^{-1}.$$

We note that this equation is identical to that obtained in the previous section with an adjusted time discount rate ($\rho\pi_{HL}$). It follows that the solution is

$$Q_H = \Lambda_H F_H \operatorname{diag}(\eta_{H,i}) F_H^{-1},$$

where $(\ell_{H,i}, F_H)$ diagonalize the matrix $\Lambda_H^{-1}\Sigma_H = F_H \operatorname{diag}(\ell_{H,i})F_H^{-1}$and the $\eta_{H,i}$ are given by:

$$\eta_{H,i} = \frac{\rho\pi_{HL} - 1 - \ell_{H,i}\gamma + \sqrt{(\rho\pi_{HL} - 1 - \ell_{H,i}\gamma)^2 + 4\ell_{H,i}\gamma\rho\pi_{HL}}}{2\rho\pi_{HL}}.$$

We can calculate the optimal trading speeds and the aim portfolios in both states. As discussed earlier, in the L-state where transaction costs are zero, it is optimal to move instantaneously to the aim portfolio, that is, $\tau_L = I$. In contrast, in the high transaction cost state H, it is optimal to trade slowly, with a trading speed $\tau_H = F_H \operatorname{diag}(\eta_{H,i})F_H^{-1}$, towards the aim portfolio. The aim portfolio in the high transaction cost state H is the conditional Markowitz portfolio, that is, $aim_H = m_H = (\gamma\Sigma_H)^{-1}\mu_H$. Intuitively, in the state H, the aim portfolio does not take into account the investment opportunity set in the zero-transaction cost state

L, because when the economy transitions to state L the investor can immediately rebalance to the first best position at zero cost. However, in the zero-transaction cost state, the aim portfolio is a linear combination of the two Markowitz portfolios m_H and m_L: $aim_L = (I - \alpha_L)m_L + \alpha_L m_H$, where the weight put on the H-state Markowitz portfolio is $\alpha_L = [\gamma\Sigma_L + \rho\pi_{LH}Q_H]^{-1}\rho\pi_{LH}Q_H$. To summarize, when there are no transaction costs in the low state the optimal trading strategy is:

$$
\begin{aligned}
n_{H,t} &= (I - \tau_H)n_{t-1} + \tau_H m_H \\
\tau_H &= F_H \operatorname{diag}(\eta_{H,i})F_H^{-1} \\
n_{L,t} &= aim_L = (I - \alpha_L)m_L + \alpha_L m_H \\
\alpha_L &= [\gamma\Sigma_L + \rho\pi_{LH}Q_H]^{-1}\rho\pi_{LH}Q_H.
\end{aligned}
$$

2.3. The case with $\Lambda_L > 0$ and $\Lambda_H = \infty$

We now consider the polar case, where transaction costs are infinite in the H-state. Clearly, it is then optimal not to rebalance in the high state. Following the derivation of our model, with no rebalancing in the H-state, we see that the equation for Q_H simplifies to:

$$Q_H = \gamma\Sigma_H + \rho\bar{Q}_H.$$

In turn, this implies that the equation for Q_L becomes:

$$I - \Lambda_L^{-1}Q_L = [\gamma\Lambda_L^{-1}(\Sigma_L + \frac{\rho\pi_{LH}}{1-\rho\pi_{HH}}\Sigma_H) + I + \rho_L\Lambda_L^{-1}Q_L]^{-1}$$

with $\rho_L = \rho(\pi_{LL} + \frac{\rho\pi_{LH}}{1-\rho\pi_{HH}})$. This equation admits an explicit solution as before, in terms of the diagonalization of the matrix $\Lambda_L^{-1}(\Sigma_L + \frac{\rho\pi_{LH}}{1-\rho\pi_{HH}}\Sigma_H) = F_L \operatorname{diag}(\ell_{L,i})F_L^{-1}$.

It follows that the solution is $Q_L = \Lambda_L F_L \operatorname{diag}(\eta_{L,i})F_L^{-1}$ where the $\eta_{L,i}$ are given by:

$$\eta_{L,i} = \frac{\rho_L - 1 - \ell_{L,i}\gamma + \sqrt{(\rho_L - 1 - \ell_{L,i}\gamma)^2 + 4\ell_{L,i}\gamma\rho_L}}{2\rho_L}.$$

In this case the optimal trading strategy is:

$$
\begin{aligned}
n_{H,t} &= n_{t-1} \\
n_{L,t} &= (I - \Lambda_L^{-1}Q_L)n_{t-1} + \Lambda_L^{-1}Q_L aim_L \\
aim_L &= (1 - \alpha_L)m_L + \alpha_L m_H \\
\alpha_L &= \{(1 - \rho\pi_{HH})\Sigma_H^{-1}\Sigma_L + \rho\pi_{LH}\}^{-1}\rho\pi_{LH}.
\end{aligned}
$$

To summarize, when transaction costs are infinite in state H it is clearly optimal to not rebalance in that state. Instead, in state L, both the speed of trading and the aim portfolio depend on the investment opportunity set in the H-state. The aim portfolio puts more weight on the H-conditional Markowitz portfolio the higher the probability to transition to that state (π_{LH}), the more persistent the state is (π_{HH}), and the higher the variance of returns in that state relative to the L-state ($\Sigma_H^{-1}\Sigma_L$). The trading speed on the other hand increases in both Σ_H and Σ_L as well as the persistence of the low and high states.

2.4. The general case

For the general case, we need to solve the system of coupled matrix equations (5) for (Q_H, Q_L):

$$I - \Lambda_s^{-1} Q_s = [\Lambda_s^{-1}(\gamma\Sigma_s + \rho\pi_{ss'}Q_{s'}) + I + \rho\pi_{ss}\Lambda_s^{-1}Q_s]^{-1}.$$

While we cannot solve the system in general, we observe that in the special case where the eigenvectors of the covariance and transaction cost matrices remain identical across states and only the eigenvalues change, the system does admit a simple explicit solution. This is a 'knife-edge case' in the general space of unconstrained matrices.[13] Still, it is an interesting parametrization, as it nests the special case where both the transaction cost and covariance matrices are diagonal with arbitrary coefficients in all states. It also nests the special case considered in GP where the transaction cost matrix is proportional to the covariance matrix, but here with possible state-dependent constants of proportionality (i.e., where $\Lambda_s = \lambda_s\Sigma_s$ and $\Sigma_{s'} = \beta\Sigma_s$ for some positive scalars $\beta, \lambda_s, \lambda_{s'}$). Also, for the general case of unconstrained matrices that can be solved numerically, we propose a simple and efficient algorithm to compute the solution. We summarize these results in the following lemma.

Lemma 2. *If $\Lambda_s = F\,\mathrm{diag}(\lambda_{i,s})F^{-1}$ and $\Sigma_s = F\,\mathrm{diag}(\upsilon_{s,i})F^{-1}\,\forall s = H, L$, then the solution of the system of matrix equations in Eq. (5) is $Q_s = \Lambda_s F\,\mathrm{diag}(\eta_{s,i})F^{-1}$ where $\forall i = 1, \ldots, n$ the constants $(\eta_{H,i}, \eta_{L,i})$ solve the system of coupled quadratic equations:*

$$\frac{\lambda_{i,s}}{1-\eta_{i,s}} = \gamma\upsilon_{i,s} + \rho\pi_{ss'}\eta_{i,s'}\lambda_{i,s'} + \lambda_{i,s} + \rho\pi_{ss}\eta_{i,s}\lambda_{i,s}.$$

In general, when Σ_s, Λ_s do not have identical eigenvectors across states, then the solution to the system of matrix equations in Eq. (5) can be obtained by the following recursion.

Given an initial (Q_H^{n-1}, Q_L^{n-1}), perform the eigenvalue decomposition (for $s = H, L$) of $\Lambda_s^{-1}(\gamma\Sigma_s + \rho\pi_{ss'}Q_{s'}^{n-1}) = F_s\,\mathrm{diag}(\ell_{i,s})F_s^{-1}$. Then set $Q_s^n = \Lambda_s F_s\,\mathrm{diag}(\eta_{i,s})F_s^{-1}$ where the $\eta_{i,s}$ solve the equation

$$1 - \eta_{i,s} = [\ell_{i,s} + 1 + \rho\pi_{ss}\eta_{i,s}]^{-1},$$

[13] To understand the parameter restrictions, note that given that both transaction cost and covariance matrices are symmetric positive definite they each would have $n(n+1)/2$ free parameters (subject to the restriction that they are positive definite). When we constrain all four (i.e., two in each state) matrices to have the same eigenvectors, then the total number of free parameters becomes $n(n+1)/2$ parameters for one matrix and only n parameters for the other three matrices. Indeed, since the latter matrices inherit the eigenvectors of the first matrix, each has only n free parameters, corresponding to their positive eigenvalues.

that is:

$$\eta_{i,s} = \frac{\rho\pi_{ss} - 1 - \ell_{i,s} + \sqrt{(\rho\pi_{ss} - 1 - \ell_{i,s})^2 + 4\ell_{i,s}\rho\pi_{ss}}}{2\rho\pi_{ss}},$$

and iterate until convergence. It is natural to use as an initial guess for Q_s^0 either the zero matrix, or the solution corresponding to $\pi_{ss} = 1$.

We conjecture that the algorithm will be especially useful for large numbers of stocks, where iterating over the $N(N+1)$ elements of the Q_L and Q_H matrices should be less efficient than iterating over the $2N$ diagonal $\eta_{i,s}$ elements. In our applications, we found that only three to five iterations are sufficient to achieve convergence. Given a numerical solution of the Q_H and Q_L matrices, we can analyze the optimal trading rule and aim portfolios.

3. Implications of the model

In this section, we illustrate the insights of our model using two simple numerical experiments. In the first application, we have two assets differing in their ranking of Sharpe ratios across two states of the economy. We analyze the aim portfolio and trading speeds when each asset's trading cost is state-dependent. In the second experiment, we analyze the sensitivity of the risk-parity allocation strategy to stochastic trading costs.

3.1. Example: Corporate vs. Treasury bonds

We illustrate some of the important implications of the model with a two-asset and two-state example. The two states are high- and low-volatility. The two assets are intended to capture salient features of a "Corporate" and a "Treasury" bond. The Corporate has a higher unconditional Sharpe ratio than does the Treasury, but is more expensive to trade. However, in the (low-probability) high-risk state, the Corporate's Sharpe ratio falls below that of the Treasury and becomes more expensive to trade. The realized returns of the two bonds are assumed to be positively correlated at 0.5. Note that we use these numbers for illustration only, as our model parameters are not calibrated to actual bond returns or transaction cost distributions.

[Insert Table 1 about here.]

The left panel of Table 1 provides the parameters for this example. We assume that the initial prices for two bonds are each \$100. In the low-risk state, the annualized volatility of each of the two bonds is \$10. Asset 1—the Corporate—has an annual expected price change of \$10, while the Treasury (Asset 2) has an annualized expected price change of \$8. Thus, the low-volatility state Markowitz portfolio has a larger investment in the Corporate than the Treasury.

However, when the economy transitions to the high-volatility state the annualized price-change volatilities jump to \$30, and the annualized expected price changes of the Corporate and Treasury bonds change to \$12 and \$16, respectively. In the high-volatility state, the Treasury has the higher expected return, so the Markowitz portfolio now holds more Treasuries.

[Insert Figure 1 about here.]

In Fig. 1 we plot weights on the Corporate and Treasury in the conditional Markowitz- and aim-portfolios, in both the low- and high-risk states, as a function of the price impact for the Corporate in the high-risk state. All other parameters are consistent with the left panel of Table 1, and trading is daily.

Fig. 1 shows that the aim portfolio weights in the high-risk state are always very close to the Markowitz portfolio weights. Intuitively, since our parameters are set so that the price impact is very small in the low-risk state, the aim portfolio in the high-risk state need not take into account the investment opportunity set in the low-risk state.

However, as we increase the price impact for the Corporate in the high-risk state, the aim portfolio starts to put less weight on Corporates and more in Treasuries. For high enough expected trading costs of Corporates in the H-state, it becomes optimal to hold more Treasuries even in the low-risk state. That is, it is optimal to hold more of the asset that appears dominated in Sharpe ratio terms in the L-state to preemptively anticipate the future (optimal) deleveraging in the H-state. Intuitively, you don't want to be stuck with a large position in corporates when the economy transitions to the high-risk (and high transaction cost) state where it will be extremely expensive to sell the corporates.

[Insert Figure 2 about here.]

Fig. 2 plots the corresponding trading speeds in both assets in both regimes.[14] Intuitively, we see that the trading speed is generally higher in the high-risk regime due to the higher volatility. However, as it becomes more costly to trade Corporates in that regime, its trading speed drops rapidly. Interestingly, the trading speed of Corporates actually increases in the low-risk regime in response to the increase of its trading cost in the high-risk regime. That is, even though Corporate portfolio is more expensive to trade in the low-risk regime, it is optimal to trade it more aggressively in anticipation of its much relative trading cost in the high-risk regime.

This example captures some salient features of the Corporate versus Treasury bond returns. Corporate bonds typically offer higher expected rates of returns in expansions (good states) than Treasury bonds. However, during recessions (bad states) their risk increases dramatically and, empirically, their expected returns fall relative to Treasuries.[15] Further, corporate bonds become far more expensive to trade in recessions, while Treasuries remain liquid. As the stylized example demonstrates, because it is optimal to reduce the position in the Corporates in the high-risk state when these are very costly to trade, it can be optimal to hold a larger share of the Treasuries already in the good state even though in that state the conditional Sharpe ratio of Corporates dominates that of Treasuries. Further, even though Corporates may less liquid than Treasuries

[14] For simplicity, we only plot the diagonal values of the trading speed matrix $\Lambda_s^{-1}Q_s$, which is actually not diagonal in this example.

[15] Of course, it is arguable whether the *expected return* is actually lower, since expected returns are hard to measure. For illustration we assume that in the bad states the risk of the Corporate bond is higher and its Sharpe ratio is lower.

in the good state, it may be optimal to trade them more aggressively in the good state in anticipation their much lower liquidity in the high-risk regime.

This example provides an answer to the question: in a portfolio with liquid and illiquid assets, which one should one liquidate first because of a liquidity shock? Our analysis gives the following answer. First and foremost, one should trade the illiquid asset more aggressively in anticipation of the future liquidity crisis and steer the portfolio to a position that overweights liquid assets, possibly deviating from the unconditional optimal portfolio to take into account the future possible risk and liquidity shocks. Of course, when the crisis does hit, one should trade the less liquid asset less aggressively and the more liquid assets more aggressively in steering the portfolio towards the conditional mean-variance efficient portfolio.

3.2. A risk-parity strategy

A risk-parity asset allocation strategy attempts to maintain steady contributions to risk from different asset classes by down weighting an asset class when its risk spikes. Such strategies have received considerable attention among practitioners, and have notably been applied in the Bridgewater "All Weather" fund. A rationale for such strategies is that, if expected returns and correlations across asset classes are difficult to forecast, and are uncorrelated with measured risk, it can be optimal to size positions in these asset classes based on forecast return variance alone. For example, if expected price changes are constant and equal (e.g., $\mu = \mathbf{1}$) and all correlation coefficients equal to zero, the mean-variance efficient Markowitz portfolio becomes a 'risk-parity' portfolio in that the weight on each asset is proportional to the inverse of its variance ($m_s = (\gamma\Sigma_s)^{-1}\mathbf{1} = \mathrm{vec}(\frac{1}{\gamma\upsilon_{i,s}})$), where $\upsilon_{i,s}$ is the price-change variance of asset i in state s.[16] Here we illustrate that, in this setting, it is optimal to deviate from the 'risk-parity allocation' if the costs of trading the asset classes are a function of their risk, which is certainly the case empirically (Almgren et al., 2005). With $\Sigma_s = \mathrm{diag}(\upsilon_{i,s})$ and $\Lambda_s = \mathrm{diag}(\lambda_{i,s})$ and $\mu_s = \mathbf{1}$, we can solve for the optimal aim portfolio in closed-form from Lemma 3 with $F = \mathrm{diag}(\mathbf{1})$.

The right panel of Table 1 provides the parameters for the risk-parity example. The initial prices for the "safe" and "risky" assets, Assets 1 and 2, respectively, are each \$100. In the low-risk state the annualized price-change volatilities are \$10 and \$30. But when the economy transitions to the high-volatility state the volatilities jump to \$20 and \$60, respectively. For each asset, the expected annualized price change is \$1 in both states.

[Insert Figure 3 about here.]

[16] An alternative 'risk-parity' strategy is to size the position in each asset to be inversely related to its standard deviation rather than its variance as we do here. This can be rationalized in a mean-variance framework by assuming that all return correlation coefficients are zero and that relative Sharpe ratios are identical across asset classes, i.e., $\frac{\mu_{i,s}}{\sigma_{i,s}} = c_s$ for every asset class i. In that case the mean-variance efficient portfolio becomes $m_s = (\gamma\Sigma_s)^{-1}\mu_s = c_s\,\mathrm{vec}(\frac{1}{\gamma\sigma_{i,s}})$. See Asness, Frazzini and Pedersen (2012) for further discussion. The insights we develop in our example apply to any type of 'risk-based' asset allocation that implies deleveraging of the more volatile assets when risk increases.

We illustrate in Fig. 3 how the aim portfolio in state s starts to deviate significantly from the risk parity portfolio as the transaction costs in state s' increase. We see that, in the high-risk state, the aim portfolio remains very close to the risk-parity portfolio even for large values of η. Intuitively, the aim portfolio in the high-risk state is close to the Markowitz portfolio because, when the economy transitions to the low-risk state, you can quickly trade out of that portfolio and towards the optimal portfolio. It is only in the low-risk state that it is optimal to deviate significantly from the risk-parity weights, because there you must anticipate the transition to the high-risk state where transaction costs are large. Indeed, in the low-risk state, the aim-portfolio weights for both assets in are lower than their weights in the Markowitz portfolio, and these weights decrease with increasing cost of trading in the high-risk state.

[Insert Figure 4 about here.]

Trading speeds for all assets are plotted in Fig. 4. As we can see, trading speed decreases in the high-risk state and increases in the low-risk state when transaction costs in the high-risk state are increasing. That is, the more costly it becomes to trade assets in the H-state, the more aggressively we have to trade assets in the low-risk state. We note that trading speeds are not security specific in this experiment, because we assume that the price impact matrix is a constant multiple of the covariance matrix.

4. A Regime Switching Model for Returns

Following much of the literature (e.g., GP; Litterman, 2005) the model presented in Section 2 assumes that, conditional on a state, the covariance matrix of *price changes* is constant. This leads to a very tractable solution because the resulting conditional aim portfolio is constant in the number of shares of each asset, and thus is independent of any changes in the prices of these assets. Thus, until there is a transition to a new state, once an investor has traded to the aim portfolio, she won't need to rebalance this portfolio when prices change. Unfortunately the assumption of a constant price-change covariance matrix, which results in a Gaussian-normal distribution for prices, is both implausible for common stock returns—as such a model permits prices to fall below zero—and is inconsistent with the data. Empirically, returns are much better described by a conditionally log-normal distribution. Fortunately, in our framework a log-normal model, that is one in which the conditional expected return and return covariance matrix is constant in a given state, is very tractable. In this section, we present a regime-switching model formulated in returns and dollar-holdings as opposed to price-changes and number of shares. In our empirical analysis in Section 5 we apply this model to timing the market portfolio while accounting for time-varying transaction costs and stochastic volatility.

4.1. Formulation

We have N risky assets and collect the N-dimensional vector of returns from period t to $t+1$ in $r_{t+1} \equiv \frac{dS_t}{S_t}$. The net return vector has the following state-dependent mean and covariances:

$$\begin{aligned} \mathrm{E}[r_{t+1}] &= \mu(s_t) \\ \mathrm{E}[(r_{t+1} - \mu(s_t))(r_{t+1} - \mu(s_t))^\top] &= \Sigma(s_t), \end{aligned}$$

where $\mu(s_t)$ and $\Sigma(s_t)$ are, respectively, the N-vector of expected returns and the $N \times N$ covariance matrix of returns. Both μ and Σ are a function of a state variable s_t which follows a Markov chain with transition probabilities $\pi_{s,s'}$.

Since the model is set up in dollars, the investor rebalances at the end of each period again in dollars. If the dollar trade vector is given by u_t, then the dollar holdings of the investor have the following dynamics:

$$\begin{aligned} x_{t+1} &= \mathrm{diag}(\mathbf{1} + r_{t+1})x_t + u_{t+1} && (8) \\ &= \mathrm{diag}(R_{t+1})x_t + u_{t+1}, && (9) \end{aligned}$$

where the gross returns are given by R_{t+1}.

We consider the optimization problem of an agent with the following objective function with an infinite investment horizon:[17]

$$\max_{x_t} \mathrm{E}\left[\sum_{t=1}^{\infty} \rho^{t-1} \left\{x_t^\top \mu(s_t) - \frac{1}{2}\gamma x_t^\top \Sigma(s_t) x_t - \frac{1}{2} u_t^\top \Lambda(s_t) u_t\right\}\right]. \quad (10)$$

The agent chooses her dollar holdings x_t in each period t so as to maximize this objective function. Specifically, at the end of period $t-1$, the agent holds x_{t-1} dollars. At this point the agent observes the state s_t, and trades u_t dollars to bring his dollar holdings to $\mathrm{diag}(R_t)x_t + u_t$. We again consider a linear price impact model. The total (dollar) cost of trading u_t is $\frac{1}{2}u_t^\top \Lambda(s_t) u_t$.

4.2. Value functions and optimal portfolio

For simplicity, we consider a two-state Markov chain model, with states H and L. The model is straightforward to generalize to multiple states. In our empirical application in Section 5 we consider two-state and four-state models. Using the dynamic programming principle, the value function $V(x_{t-1}, R_t, s_t)$ satisfies

$$V(x_{t-1}, R_t, s) = \max_{x_t} \left(x_t^\top \mu_s - \frac{1}{2} u_t^\top \Lambda_s u_t - \frac{\gamma}{2} x_t^\top \Sigma_s x_t + \rho \mathbb{E}_t \left[V\left(x_t, \mathbf{1} + \mu_s + \epsilon_s, z\right)\right] \right),$$

[17] In the Internet Appendix, as discussed in footnote 9 we provide two ways to micro-found this objective function.

where $\mathbb{E}[\epsilon_s] = 0$ and $\mathbb{E}\left[\epsilon_s \epsilon_s^\top\right] = \Sigma_s$. We guess the following quadratic form for our value functions:

$$V(x, R, s) = -\frac{1}{2} x^\top \operatorname{diag}(R) Q_s \operatorname{diag}(R) x + x^\top \operatorname{diag}(R) q_s + c_s,$$

where Q_s is a symmetric $N \times N$ matrix and q_s, c_s are N-dimensional vectors of constants for $s \in \{H, L\}$. We can now simplify $\mathbb{E}_t\left[V\left(x_t, \mathbf{1} + \mu_s + \epsilon_s, z\right)\right]$ using the assumed structure for the value functions and write it in the form of $-\frac{1}{2} x_t^\top A_s x_t + x_t^\top b_s + d_s$ where

$$\begin{aligned}
Z_s &= \mathbb{E}[\left(\mathbf{1} + \mu_s + \epsilon_s\right)\left(\mathbf{1} + \mu_s + \epsilon_s\right)^\top] = \Sigma_s + \left(\mathbf{1} + \mu_s\right)\left(\mathbf{1} + \mu_s\right)^\top, \\
A_s &= \pi_{s,s}(Z_s \circ Q_s) + \pi_{s,s'}(Z_s \circ Q_{s'}), \\
b_s &= \pi_{s,s}(\mu_s \circ q_s) + \pi_{s,s'}(\mu_s \circ q_{s'}), \\
d_s &= \pi_{s,s} c_s + \pi_{s,s'} c_{s'},
\end{aligned}$$

and $\circ$ denotes element-wise multiplication. Using this expression for $\mathbb{E}_t\left[V\left(x_t, \mathbf{1} + \mu_s + \epsilon_s, z\right)\right]$, we obtain

$$\begin{aligned}
V(x_{t-1}, R_t, s) = \max_{x_t} \Big\{ & x_t^\top \mu_s - \frac{1}{2}\left(x_t - \operatorname{diag}(R_t) x_{t-1}\right)^\top \Lambda_s \left(x_t - \operatorname{diag}(R_t) x_{t-1}\right) - \frac{\gamma}{2} x_t^\top \Sigma_s x_t \\
& - \frac{\rho}{2} x_t^\top A_s x_t + \rho x_t^\top b_s + \rho d_s \Big\}.
\end{aligned}$$

Thus, we maximize the quadratic objective $-\frac{1}{2} x_t^\top J_s x_t + x_t^\top j_t^s + k_s$ where we define

$$\begin{aligned}
J_s &= \gamma \Sigma_s + \Lambda_s + \rho A_s \\
j_s &= \Lambda_s \operatorname{diag}(R_t) x_{t-1} + \mu^s + \rho b_s \\
k_s &= -\frac{1}{2} x_{t-1}^\top \operatorname{diag}(R_t) \Lambda_s \operatorname{diag}(R_t) x_{t-1} + \rho d_s.
\end{aligned}$$

Then, the optimal x_t when the state is s is given by $J_s^{-1} j_s$. That is to say

$$x_t = \left(\gamma \Sigma_s + \Lambda_s + \rho A_s\right)^{-1} \left(\Lambda_s \operatorname{diag}(R_t) x_{t-1} + \mu^s + \rho b_s\right). \tag{11}$$

The value achieved at the optimal solution is given by $\frac{1}{2} j_s^\top J_s^{-1} j_s + k_s$ and we obtain the following coupled matrix equations:

$$Q_s = -\Lambda_s \left(\gamma \Sigma_s + \Lambda_s + \rho A_s\right)^{-1} \Lambda_s + \Lambda_s, \tag{12}$$

$$q_s = \Lambda_s \left(\gamma \Sigma_s + \Lambda_s + \rho A_s\right)^{-1} \left(\mu_s + \rho b_s\right), \tag{13}$$

$$c_s = \frac{1}{2}(\mu_s + \rho b_s)^\top \left(\gamma \Sigma_s + \Lambda_s + \rho A_s\right)^{-1} \left(\mu_s + \rho b_s\right) + \rho d_s. \tag{14}$$

Overall, these equations are very similar to those obtained in the previous section for the regime-switching model of price changes. The main difference is the need to introduce the matrices A_s and b_s which are nonlinear transformations of Q_s and q_s. We solve for Q_s and q_s iteratively from Eqs. (12) and (13), respectively. We use the zero matrix for Q_s and the zero vector for q_s as initial guesses. Convergence is obtained very rapidly in all of our implementations.

4.3. Aim portfolio and trading speed

Following our analysis in the previous section, we define the aim portfolio in each state, aim_s, as the portfolio at which it would be optimal not to rebalance given the current state s. The following lemma characterizes the aim portfolio and the trading speed.

Lemma 3. *The conditional aim portfolio aim_s at which it is optimal not to rebalance is given by*

$$aim_s = \left(\gamma\Sigma_s + \rho A_s\right)^{-1}\left(\mu_s + \rho b_s\right).$$

It maximizes the value function $V(x_{t-1}, R_t, s)$ with respect to $x_{t-1}\,\mathrm{diag}(R_t)$.

The optimal trading rule is to "trade partially towards the aim" at the trading speed $\tau_s = \Lambda_s^{-1}Q_s$:

$$x_s = (I - \tau_s)\,\mathrm{diag}\left(R_t\right)x_{t-1} + \tau_s aim_s.$$

Proof. Maximizing the value function at time $V(x_{t-1}, R_t, s)$ with respect to $\mathrm{diag}\left(R_t\right)x_{t-1}$ we obtain:

$$aim_s = Q_s^{-1}q_s.$$

Substituting from the definitions in Eqs. (12) and (13) we obtain:

$$\begin{aligned}
aim_s &= \left(-\Lambda_s\left(\gamma\Sigma_s + \Lambda_s + \rho A_s\right)^{-1}\Lambda_s + \Lambda_s\right)^{-1}\left(\Lambda_s\left(\gamma\Sigma_s + \Lambda_s + \rho A_s\right)^{-1}\left(\mu_s + \rho b_s\right)\right)\\
&= \left(-\left(\gamma\Sigma_s + \Lambda_s + \rho A_s\right)^{-1}\Lambda_s + I\right)^{-1}\left(\gamma\Sigma_s + \Lambda_s + \rho A_s\right)^{-1}\left(\mu_s + \rho b_s\right)\\
&= \left(\gamma\Sigma_s + \rho A_s\right)^{-1}\left(\mu_s + \rho b_s\right)
\end{aligned}$$

where the last equality obtains by noting that if we define the matrix

$$M = \left(-\left(\gamma\Sigma_s + \Lambda_s + \rho A_s\right)^{-1}\Lambda_s + I\right)^{-1}\left(\gamma\Sigma_s + \Lambda_s + \rho A_s\right)^{-1}$$

then

$$M^{-1} = \left(\gamma\Sigma_s + \Lambda_s + \rho A_s\right)\left(-\left(\gamma\Sigma_s + \Lambda_s + \rho A_s\right)^{-1}\Lambda_s + I\right) = \left(\gamma\Sigma_s + \rho A_s\right),$$

which immediately implies that $M = \left(\gamma\Sigma_s + \rho A_s\right)^{-1}$.

To prove the second part of the lemma, we start from the definition of the optimal position x_t given in Eq. (11). It is straightforward to obtain the optimal trade

$$x_t - \text{diag}(R_t)x_{t-1} = \Lambda_s^{-1}(\gamma\Sigma_s + \rho A_s)(aim_s - x_t).$$

Using the definition of matrix M above and Eq. (12), we obtain the formula for the trading speed. □

4.4. Difference between two models

[Insert Figure 5 about here.]

Fig. 5 compares aim portfolios and trading speeds in models set up in shares (i.e., conditionally normal price distributions) and dollars (i.e., conditionally log-normal price distributions) for an example with a single risky asset, and with high and low volatility states H and L. Table 2 displays all of the model parameters. We calibrate the model to an initial share price of one dollar so that the y-axis represents both the dollar investment and the number of shares in the aim portfolio.

For the "shares" model of this section, the aim portfolio weight on the risky asset is smaller than in the "dollars" model of Section 2, in which price changes are conditionally normally distributed. Moreover, this difference increases with the expected return on the risky asset. The intuition underlying this finding is that, conditional on remaining in the same state, the dollar amount in the risky asset is constant. Thus, following positive returns, some amount of the risky asset must be sold off to rebalance to the aim portfolio. The lower position in the risky asset anticipates this future costly rebalancing by holding a lower position in the risky asset when the risky asset is expected to perform well.

[Insert Table 2 about here.]

We also observe that the trading speed is higher for the regime-switching model of returns than for that of price changes. This is because, in the regime-switching model of returns, there is an additional "rebalancing motive" for trading, as dollar positions drift away from their target as a result of return shocks (even in the absence of any change in the investment opportunity set).

5. Empirical application

In this section, we implement our methodology using the modeling framework in dollars and illustrate that there are economically significant benefits using our approach both in-sample and out-of-sample.

5.1. Model Calibration

We use daily value-weighted market returns of all firms in Center for Research in Security Prices (CRSP) from 1967-03-13 to 2017-03-31 to estimate a regime-switching model.[18] The data are downloaded from Ken

[18] This time period corresponds to 50 years of data when each year is assumed to have 252 trading days.

French's data library.

Guidolin and Timmermann (2006) consider a range of values for the number of states and find that a four-state regime model performs better in explaining bond and stock returns. Following this study, we estimate a Markov switching model with four states to describe the dynamics of market returns:

$$r_{t+1} = \mu(s_t) + \sigma(s_t)\epsilon_{t+1} \tag{15}$$

where $s_t = \{1, 2, 3, 4\}$ and ϵ_{t+1} are serially independent and drawn from standard normal distribution. State transitions occur according to a Markov chain and we denote by P_{ij} the probability of switching from state i to state j.[19]

[Insert Table 3 about here.]

Table 3 displays the estimates of the model.[20] All coefficients are statistically significant at 1% level. Overall, we observe that the rank correlation between the estimated expected returns and volatilities is not equal to one. We observe that the expected return can be lower in a high-volatility state. This pattern has been found since the initial applications with regime switches on equity returns (see, e.g., Hamilton and Susmel, 1994).

[Insert Figure 6 about here.]

For each regime i, Fig. 6 illustrates the probability that the trading day t is in regime i conditional on the full return sample. These probabilities are referred to as "smoothed" probabilities in the regime-switching literature (Kim, 1994).[21] The first regime highlights the good states of the return data with high return and low volatility corresponding to the highest Sharpe ratio. This regime has also the highest expected duration with roughly 51 trading days. The transition from this state usually occurs to the second state with slightly lower expected return and higher volatility. The expected duration for this state is 30 trading days. The third state is a distressed state with low expected return and high volatility. This state has the lowest Sharpe ratio and has an expected duration of approximately 33 trading days. The final state covers the crisis periods with very high expected return and very high volatility. We observe that it covers trading days around the 1987 crash, the dot-com bubble, and the financial crisis. This state is relatively short-lived with an expected duration of roughly 15 trading days.

[19] To restrict the number of parameters, we have also tried fitting a four-state model that constrains the general model to having only two mean and volatility coefficients (i.e., mean or volatility may remain unchanged after a transition) as opposed to four but this constrained model can be rejected with a likelihood test.

[20] We use the MS_Regress toolbox in Matlab for the estimation of the model (Perlin, 2015).

[21] In the Internet Appendix, Fig. A.1 illustrates the color-coded regime of each trading day by identifying the state with the highest probability.

5.2. Calibration of the transaction costs

To calibrate the transaction cost multipliers of our model realistically, we use proprietary execution data from the historical order databases of a large investment bank. The orders primarily originate from institutional money managers who would like to minimize the costs of executing large amounts of stock trading through algorithmic trading services. The data consist of two frequently used trading algorithms, volume-weighted average price (VWAP) and percentage of volume (PoV). The VWAP strategy aims to achieve an average execution price that is as close as possible to the volume-weighted average price over the execution horizon. The main objective of the PoV strategy is to have constant participation rate in the market along the trading period.

[Insert Table 4 about here.]

The execution data cover Standard & Poors (S&P) 500 stocks between January 2011 and December 2012. Execution duration is greater than five minutes but no longer than a full trading day. Total number of orders is 81,744 with an average size of approximately $1 million. The average participation rate of the order, the ratio of the order size to the total volume realized in the market, is approximately 6%. Table 4 reports further summary statistics on the large-order execution data.

According to our quadratic transaction cost model, trading q dollars in state j would cost the investor $\lambda_j q^2$. Since each of our executions are completed in a day, we can uniquely label each execution originating in one of the four states by setting it to the state with maximal smoothed probability. With this methodology, we find that 22,946 executions are in regime 1, 41,898 executions in regime 2, 14,502 executions in regime 3, and 2,398 in regime 4. Compared to other states, regime 4 has a relatively small number of executions due to its short-lived nature. At first sight, it is surprising that we have the largest number of executions in regime 2. But, during the 2011—2012 period, the volatility was relatively high so there are actually fewer trading days in regime 1.

Our execution data have information on both the order size and total trading cost. Total trading cost is computed by comparing the average price of the execution to the prevailing price in the market before the execution starts. This is usually referred to as implementation shortfall (IS) (Perold, 1988). Formally, IS of the ith execution is given by

$$IS_i = \text{sgn}\left(Q_i\right) \frac{P_i^{\text{avg}} - P_{i,0}}{P_{i,0}}, \tag{16}$$

where Q_i is the dollar size of the order (negative if a sell order), P_i^{avg} is the volume-weighted execution price of the parent-order, and $P_{i,0}$ is the average of the bid and ask price at the start-time of the execution. Thus, total trading cost in dollars is equal to $IS_i \times Q_i$. According to our model, this is given by $\lambda_{m(i)} Q_i^2$ where $m(i)$ maps the ith execution to the state of the trading day. Thus, we can estimate λ_j for each state by fitting the following model:

$$IS_i = \lambda_1 Q_i \mathbf{1}_{\{m(i)=1\}} + \lambda_2 Q_i \mathbf{1}_{\{m(i)=2\}} + \lambda_3 Q_i \mathbf{1}_{\{m(i)=3\}} + \lambda_4 Q_i \mathbf{1}_{\{m(i)=4\}} + \varepsilon_i.$$

[Insert Table 5 about here.]

Table 5 illustrates the estimated coefficients. The reported standard errors are clustered at the stock and calendar day levels. We observe that λ estimates are all highly significant (except in state 4 where we observe fewer executions in our data set) and vary a lot across regimes and tend to increase with volatility. We find that λ_3 is the largest across all states. Recall that this state has the lowest Sharpe ratio and thus can be interpreted as the distressed state. Using Wald tests pairwise, we find that the estimate of transaction costs in this distressed state, λ_3, is statistically higher than all other coefficients at a 10% significance level.

[Insert Table 6 about here.]

To better understand the variation in transaction costs across our states, we present in Table 6 the average values of various liquidity proxies in each state. We find that bid-ask spreads, mid-quote volatility, and turnover are increasing across states, i.e., volatility. However, the Amihud illiquidity proxy returns similar ranking to the estimated λ coefficients with state 3 being more illiquid than state 4. Since volume is much larger in that state, it may act as a mitigating factor on trading costs (see, e.g., Admati and Pfleiderer, 1988; Foster and Viswanathan, 1993).

Since we would like to estimate the price impact of trading the market portfolio, our estimates may be overestimating the cost as it is based on the complete set of S&P 500 stocks. In order to address this issue, we rerun our regressions only using data corresponding to the top 10% of stocks with respect to market capitalization. We believe that this universe of stocks reflects a more natural comparison to the market portfolio.

The second column of Table 5 illustrates the estimated coefficients for this liquid universe. We observe that the coefficients are lower by a factor between two and three but preserve the same ranking across states. In this case, λ_3 is statistically different than the coefficients of the first and second state at 10% significance level. The second panel of Table 6 illustrates the average values of each liquidity proxy in each regime using this universe of large-cap stocks.

5.3. Objective function

We use the regime-switching model based on dollar holdings and returns presented in Section 4 as the investment horizon is very long. Formally, the investor's objective function is:

$$\mathrm{E}\left[\sum_{t=0}^{\infty} \rho^t \left\{ x_t \mu(s_t) - \frac{1}{2}\lambda(s_t) u_t^2 - \frac{\gamma}{2}\sigma^2(s_t) x_t^2 \right\}\right] \tag{17}$$

where $x_t = x_{t-1}(1 + r_t) + u_t$ and $s_t \in \{1, 2, 3, 4\}$. We calibrate ρ so that the annualized discount rate is 1%. We set $\gamma = 1 \times 10^{-10}$ which we can think of as corresponding to a relative risk aversion of 1 for an agent with $10 billion dollars under management. We assume that the investor starts from zero holdings and rebalances daily.

The optimal portfolio policy of the investor is given by

$$x_t^{\text{opt}}(s_t) = (1 - \frac{Q(s_t)}{\lambda(s_t)})(1 + r_t)x_{t-1}^{\text{opt}} + \frac{Q(s_t)}{\lambda(s_t)} aim(s_t) \qquad \forall s_t \in \{1, 2, 3, 4\} \tag{18}$$

where q and Q solve the following system of equations $\forall s \in \{1, 2, 3, 4\}$:

$$Q(s_t) = -\lambda(s_t)^2 \left(\gamma\sigma^2(s_t) + \lambda(s_t) + \rho\left(\sigma^2(s_t) + (1 + \mu(s_t))^2\right)\overline{Q}(s_t)\right)^{-1} + \lambda(s_t), \tag{19}$$

$$q(s_t) = (\mu(s_t) + \rho\mu(s_t)\overline{q}(s_t))\left(1 - \frac{Q(s_t)}{\lambda(s_t)}\right), \tag{20}$$

$$aim(s_t) = Q(s_t)^{-1}q(s_t). \tag{21}$$

Since we have only one asset, the trading speed is one-dimensional and given by $\frac{Q(s_t)}{\lambda(s_t)}$ in each state.

5.4. Aim portfolios and trading speed

[Insert Figure 7 about here.]

Using the estimated model coefficients, we first study the aim portfolios across states in the presence and absence of transaction costs. Fig. 7 illustrates the aim portfolios for the optimal policy in these cases. We also compare this optimal policy with a simple unconditional mean-variance benchmark, in which the portfolio rule holds a constant dollar amount equal to $\frac{\mu_{\text{avg}}}{\gamma\sigma^2_{\text{avg}}}$ in the risky asset. Here, μ_{avg} and σ^2_{avg} are the sample mean and variance of the market returns from 1967-03-13 to 2017-03-31.

In the left panel, the solid line illustrates the aim portfolios in the absence of transaction costs. Without transaction costs, aim portfolios are simply the conditional mean-variance optimal Markowitz portfolios. Compared to the unconditional mean-variance constant benchmark portfolio, the conditional Markowitz portfolio is very aggressive in regime 1 and holds a smaller amount than the constant portfolio in all other states. In regime 3, the holdings are very close to a risk-free position.

In the right panel, we plot the aim portfolios when there are stochastic trading costs. We use the estimated transaction cost multipliers from the liquid subset as provided in Table 5. Surprisingly, regime 4 has the smallest aim portfolio whereas regime 3, the lowest Sharpe ratio state, has slightly higher holdings. This is due to differences in trading costs, as well as to the transition probabilities, across states. For example, trading costs are largest in regime 3, thus the optimal aim portfolio, which will determine trading in that state, should depend on the average positions expected in states that it will transition from, essentially regime 4 (probability of $\approx 6\%$) and regime 2 (probability of $\approx 1\%$), as well as from states it will transition to, again regime 2 (probability of $\approx 2\%$) and regime 4 (probability of $\approx 1\%$). These considerations make the desired holdings in regime 3 higher. Interestingly, the aim portfolios in regime 3 and regime 4 hold a larger position in risky assets than the corresponding conditional Markowitz portfolios, whereas the aim portfolio in regime 1 actually holds a much smaller position than the conditional Markowitz portfolio. This emphasizes the impact of transaction costs and potential transitions between states on desired holdings.

[Insert Figure 8 about here.]

Finally, we plot the trading speeds in each regime in Fig. 8. Due to high volatility, regime 4 has the highest trading speed. Regime 1 has the lowest trading costs so we find that the trading speed is relatively larger compared to regime 2 and regime 3. However, the difference is not very large as these other states have higher volatilities. Regime 3 has the lowest trading speed potentially due to its highest trading costs.

5.5. In-sample analysis

In this section, we evaluate the performance of the optimal policy using the in-sample estimates from our four-state regime-switching model. We compare it to various benchmark policies in the presence and absence of transaction costs to help to develop an understanding of the potential benefits of this methodology. In Section 5.7, we repeat this exercise with a full out-of-sample analysis to better quantify these benefits.

In order to evaluate the performance of the policies, we need to assign each trading day to a regime state so that we can determine the appropriate values of $\sigma^2(s_t)$ and $\lambda(s_t)$. For this purpose, we use the smoothed probabilities from the regime-switching model and assign the regime of each trading day to the state with the highest smoothed probability. We skip a day to implement the optimal and myopic policies. That is to say, to determine the position on day t, we use the smoothed probabilities from day $t-1$.

Let x_t^{opt} be the optimal policy as computed from Eq. (18) and the above implementation methodology. We break down the realized objective function into two terms, wealth and risk penalties:

$$W_T^{\text{opt}} = \sum_{t=1}^{T=12600} \rho^t \left[x_t^{\text{opt}} r_{t+1} - \frac{1}{2}\lambda(s_t)\left(x_t^{\text{opt}} - x_{t-1}^{\text{opt}} R_t\right)^2 \right] \tag{22}$$

$$RP_T^{\text{opt}} = \sum_{t=1}^{T=12600} \frac{1}{2}\rho^t \left[x_t^{\text{opt}}\sigma(s_t)\right]^2. \tag{23}$$

Here, $t = 12601$ corresponds to the final trading day of 2017 Q1.

5.5.1. Benchmark policies

As described earlier, the first benchmark policy is the constant-dollar rule in which the investor chooses $x_t^{\text{con}} = \frac{c\mu_{\text{avg}}}{\gamma\sigma^2_{\text{avg}}}$. The parameters, μ_{avg} and σ^2_{avg}, are obtained using the full in-sample data. We choose c so that the policy has the same risk exposure as the optimal policy, i.e., the discounted sum of risk penalties from this policy equals RP_T^{opt}. In the presence of trading costs, getting into a large constant position in the first period may result in large trading costs, so to minimize this effect we allow this policy to build the constant position in the first ten trading days with equal-sized trades.

The second benchmark policy is the buy-and-hold portfolio in which the investor invests x_0 dollars into the market portfolio at the beginning of the horizon.[22] We provide a slight advantage to this benchmark policy by assuming that he builds this position with no trading costs. The investor never trades till the end of the investment horizon. We again optimally choose x_0 so that the policy has the same risk exposure as the optimal policy.

The third benchmark policy is the myopic policy with transaction cost multiplier, a widely used practitioner approach. This approach solves a myopic mean-variance problem, that is, given some initial position (x_{t-1}) and the state s_t, r_t, it solves $\max_{u_t} x_t\mu(s_t) - \frac{1}{2}\gamma\sigma(s_t)^2 x_t^2 - \frac{1}{2}hu_t^2\lambda(s_t)$ subject to the dynamics $x_t = x_{t-1}(1+r_t)+u_t$. The myopic policy with transaction cost multiplier h is given by

$$x_t^{\text{my}}(s_t) = (1-\tau(s_t))(1+r_t)x_{t-1}^{\text{my}} + \tau(s_t)\frac{\mu(s_t)}{\gamma\sigma^2(s_t)} \qquad \forall s_t \in \{1,2,3,4\} \tag{24}$$

$$\tau(s_t) = \frac{1}{1+\frac{h\lambda(s_t)}{\gamma\sigma^2(s_t)}}. \tag{25}$$

Note that this policy, like the optimal one, trades partially towards an aim portfolio. However, since it takes the current state as given and ignores the implications of any future transitions in the state, the aim portfolio is the conditional mean-variance efficient Markowitz portfolio and the trading inertia, $1-\tau(s_t) \approx \frac{h\lambda(s_t)}{\gamma\sigma^2(s_t)}$, only depends on the ratio between current state's transaction costs and the variance. We choose h so that the myopic policy uses the optimal trading speed $\tau^*(s_t)$ in each regime. Note that in this case, the risk penalties will not be the same. Further, in the absence of transaction costs, the myopic policy is optimal, thus, we compare it to the optimal one only in the presence of transaction costs.

5.5.2. Comparison between portfolio policies

[Insert Figure 9 about here.]

Fig. 9 compares the optimal policy to the constant portfolio in the absence of trading costs. Both policies have the same risk penalty by construction (see bottom-right panel), thus the wealth dynamics are direct measures of performance. The top-left panel illustrates that the optimal policy has a much higher performance. We observe that this is achieved by trading more and timing the regimes of the return data. This confirms that there is predictability and that, at least in the absence of transaction costs, there is value to rebalancing across the estimated regimes.

[Insert Figure 10 about here.]

Fig. 10 compares the optimal policy to the buy-and-hold portfolio in the absence of trading costs. Both policies again have the same risk penalty by construction. In the top-right panel, the starting position for the buy-and-hold policy is roughly $\$3 \times 10^9$. Since this policy never trades, the position becomes very large at

[22] We assume that the investor shorts the risk-free asset to generate this initial capital so he also starts from zero wealth.

the end of the horizon which causes this policy to take much higher risk. This policy performs worse than the constant portfolio for that reason. Since there are no trading costs, the constant portfolio maintains the same level of position costlessly and manages the risk exposure better.

[Insert Figure 11 about here.]

Fig. 11 compares the optimal policy to the constant portfolio in the presence of trading costs. Both policies again have the same risk penalty by construction. Top-left panel illustrates that the difference in performance is more pronounced in the presence of trading costs. One reason for this is the excessive trading of the constant portfolio policy as illustrated in the medium-left and bottom panels. Compared to the previous case, we note that the optimal policy trades much more slowly as shown in the medium-right panel. The constant policy trades a lot after large return shocks in order to keep a constant dollar amount invested in the market portfolio. Therefore, the constant-dollar policy incurs much larger cumulative transaction costs than the optimal policy as we see in the bottom panel, which contributes a significant portion of the observed wealth difference between the two strategies.

[Insert Figure 12 about here.]

Fig. 12 compares the optimal policy to the buy-and-hold portfolio in the presence of trading costs. They both have the same risk penalty by construction. In the top-right panel, the starting position for the buy-and-hold policy is roughly $\$1.4 \times 10^9$. This policy performs better than the constant portfolio in this case as it never incurs trading costs. We note that the buy-hold policy is very slowly moving in building the position as it can never get out of the position to manage risk. This becomes the main driver of underperformance compared to the optimal policy.

[Insert Figure 13 about here.]

Finally, Fig. 13 compares the optimal policy to the myopic policy with transaction cost multiplier. Both policies have the same trading speeds but different aim portfolios. Since the risk penalties are not the same, wealth dynamics are not the main performance metric in this case. For this reason, we also include the cumulative objective value which equals the difference between wealth and risk penalties. The performance difference as illustrated by objective values in the bottom-right panel is again substantial. The main driver seems to be excessive trading of the myopic policy. Since the myopic portfolio uses the conditional Markowitz portfolio as its aim position, it ends up trading a lot especially in the good state. Taking large positions, it also induces large risk penalties. This example shows the importance of accounting for the future dynamics of the state variables as this generates the difference between the aim portfolios of both policies.

5.6. Large vs. small portfolios

Managing transaction costs effectively will be very important when the portfolio size is large. In the absence of transaction costs, we know that the myopic portfolio, i.e., the conditional Markowitz portfolio, is optimal. Therefore, when the portfolio size is small, the difference between the optimal policy in the presence of transaction costs and the myopic portfolio may be very small. Since we are using realistic parameters, our model can also speak to the level of portfolio size at which managing transaction costs would provide significant benefits. For example, with $\gamma = 1 \times 10^{-10}$ we observe that our aim portfolios range from approximately \$20 billion to \$85 billion dollars.

[Insert Figure 14 about here.]

Fig. 14 compares the optimal policy to the myopic policy when $\gamma = 1 \times 10^{-5}$. In this case, the top-right panel tells us that the maximum aim portfolio across states is roughly \$2.8 million and in this case, there is no significant difference between performances.

[Insert Figure 15 about here.]

Fig. 15 compares the optimal policy to the myopic policy when $\gamma = 2.5 \times 10^{-8}$. With this calibration, the aim portfolios range from approximately \$20 million to \$900 million. We observe that the myopic policy diverges a lot from the optimal policy by trading a lot and taking too much risk. It returns negative objective value and near-zero wealth levels. Thus, this simple exercise suggests that when the portfolio size is on the order of a hundred million dollars, taking price impact into account is crucial.

5.7. Out-of-sample analysis

The in-sample analysis of Section 5.5 is useful in studying the *expected* properties and benefits of a fully dynamic portfolio policy, but to better assess the value of the regime-switching model, we perform an out-of-sample analysis. We implement a two-state regime-switching model in this section for faster estimation of the parameters as we need to estimate a regime-switching model every day from 1967 to 2017.

5.7.1. Calibration

[Insert Table 7 about here.]

First, we estimate the model parameters to determine the parameters of the objective function. We use all the available market return data from 1926-07-01 to 2017-03-31. Table 7 presents the estimated coefficients. We again observe that the expected return is lower in the high-volatility state. The "good" state with higher expected return and low volatility is again more persistent.

[Insert Table 8 about here.]

We estimate the transaction cost regimes using the same methodology, but now with two regimes. We again use the estimates from the liquid subset, i.e., the 50 stocks with largest market capitalizations. Formally, we run the following regression:

$$IS_i = \lambda_1 Q_i \mathbf{1}_{\{m(i)=1\}} + \lambda_2 Q_i \mathbf{1}_{\{m(i)=2\}} + \varepsilon_i.$$

Table 8 illustrates the estimated coefficients. We observe that λ estimates are all highly significant. We find that λ_2 is greater than λ_1 and this difference is statically significant. Regime 2 has the lowest Sharpe ratio and thus can be interpreted as the distressed state.

5.7.2. Objective function

The estimated two-state regime-switching model and the calibrated transaction costs will determine the parameters of the out-of-sample objective function. Let x be any given policy. We will compute the out-of-sample performance of this policy by $W(x) - RP(x)$ where

$$W(x) = \sum_{t=1}^{T=12600} \rho^t \left[x_t r_{t+1} - \frac{1}{2}\lambda(s_t)\left(x_t - x_{t-1}R_t\right)^2 \right] \tag{26}$$

$$RP(x) = \sum_{t=1}^{T=12600} \frac{1}{2}\rho^t x_t^2 \sigma^2(s_t), \tag{27}$$

and s_t will be equal to the state with the larger smoothed probability at time t, and σ and λ will be given by the calibrations in Table 7 and Table 8 (the liquid column), respectively.

The investor is not aware of the true parameters of the model and uses only information up to trading day t in order to make a trading decision for day $t+1$, i.e., no policy will be able to use any forward looking data.

5.7.3. Optimal policy

We construct our policy based on our theoretical analysis as follows. We will label this policy as the "optimal" policy as it is based on our dynamic model. First, we estimate a two-state regime-switching model using the market return data from 1926-07-01 to 1967-03-10. We use these estimated parameters to construct a trading policy as formulated by Lemma 3. To apply our trading rule, we need to predict the regime of the next trading day. To accomplish this, we re-estimate a two-state regime-switching model using return data from 1926-07-01 to the decision date. This estimation will provide smoothed probabilities for every trading day including the decision date. We will predict the next trading day's regime using the state with the larger smoothed probability. For example, suppose that regime 1's smoothed probability for decision date is 0.52 and regime 2's smoothed probability for decision date is 0.48. We will predict the next trading day to be of regime 1.

5.7.4. Benchmark policies

We will use the constant portfolio and buy-and-hold portfolio as the benchmark policies.

We construct the constant portfolio policy in the out-of-sample data as follows. First, we estimate μ_{avg} and σ_{avg} using the market return data from 1926-07-01 to 1967-03-10. These parameters are held fixed throughout the investment horizon. The investor then constructs the following constant portfolio: $x_t^{\text{con}} = \frac{c\mu_{\text{avg}}}{\gamma\sigma_{\text{avg}}^2}$. We choose c so that the policy has the same risk exposure as the optimal policy.

The buy-and-hold portfolio is constructed similarly to its in-sample counterpart. The investor invests x_0 dollars (borrowed at the risk-free rate) into the market portfolio at the beginning of the investment horizon, i.e., on the first trading day (1967-03-13), and then never trades but cumulates returns from its risky and risk-free asset positions. We choose x_0 so that the policy has the same total risk exposure as the optimal policy.

A set of figures that illustrate the out-of sample performance of the trading rules are presented in the Internet Appendix to this paper. Figs. A.2 and A.3 compare the optimal policy to the constant portfolio and to the buy-and-hold portfolio, respectively, when trading costs are zero. The top-left panel illustrates that the optimal policy has higher performance in terms of terminal wealth. The results show that the regime-switching model captures predictability out-of sample and that it is valuable, even absent transaction costs, in timing these regimes.

Figs. A.4 and A.5 compare the optimal policy to the constant portfolio and the buy-and-hold portfolio, respectively. The top-left panel illustrates that the difference in performance is more pronounced in the presence of trading costs. The constant policy immediately rebalances to the constant weights following any return shock, resulting in large realized transaction costs and a far lower overall performance. We can see that the difference in cumulative transaction costs paid by both strategies is very large and that this difference contributes substantially to the difference in wealth generated by both strategies. This hints to an interesting insight, which we confirm in our analysis in Section 5.8: even if expected return regimes are difficult to measure, leading to a smaller out-of-sample performance in the absence of transaction costs (t-costs), if transaction cost regimes are more accurately measured, which is plausible since t-costs vary with second moments, then optimally accounting for the variation in volatility and transaction costs will lead to a sizable improvement in performance.

An interesting feature of the buy-and-hold portfolio is that it builds its position very slowly but ends up with a very large position at the end of the sample which increases the total risk. In the top-right panel of Fig. A.5, the starting position for the buy-and-hold policy is roughly $\$4.7 \times 10^9$. This is substantially lower than the aim portfolio of the optimal policy in the low-volatility state.

Overall, this out-of-sample analysis illustrates that the outperformance of the optimal policy is robust to parameter uncertainty of the regime-switching model.

5.8. Which parameter should you time?

In this section, we investigate the value of timing each switching parameter of the general model. The switching parameters are μ, σ, and λ. It is well-known, at least since Merton (1980), that expected returns are estimated less precisely than volatilities. Further, Moreira and Muir (2017) have shown that there are gains to scaling down the risky asset exposure in response to an increase in the market's variance, which suggests that the conditional mean of the market moves less than one-for-one with its variance. One might thus expect that out-of-sample the benefits of timing changes in volatility could be larger than timing changes in expected returns. We show some evidence to that effect below. Further, since transaction costs vary with volatilities, we also provide quantitative evidence about the value of timing transaction cost regimes.

We use the the implementation of the optimal policy from the out-of-sample analysis to account for the potential bias introduced by imprecisely estimated parameters. First, we study the value of timing the switches in either volatility or expected returns in the absence of trading costs. In this analysis, if the investor times volatility, he takes into account that the volatility is time-varying between two states but assumes that expected return is constant throughout the investment horizon and is given by μ_{avg} (as in the case of the constant portfolio rule). Similarly, if the investor times expected returns, he models them as time-varying between high and low states and internalizes the potential switches in the expected return in his trading rule but he assumes that the volatility stays constant at a level of σ_{avg} (as in the case of the constant portfolio rule). We scale the policies so that they take the same risk.

In the Internet Appendix, we present a set of figures that illustrate the results of this exercise. Fig. A.6 compares these two timing approaches in the absence of trading costs using an out-of-sample trading approach. We scale both policies so that they both have the same risk exposure as the optimal policy that times both parameters. We find that timing volatility provides much higher performance. The terminal wealth of the policy that only times volatility is actually higher than the terminal wealth of the optimal policy that times both parameters as shown in Fig. A.2. This illustrates that trying to time expected returns may be actually detrimental in an out-of-sample trading strategy. The top-right panel shows that the μ-timing policy has a wider range of positions compared to the range observed in the σ-timing policy. In the absence of t-costs the strategies switch to their conditional mean-variance Markowitz portolios in every state. Recall that the estimated mean in the state 2 is negative and the volatility is high. This implies that the μ-timing strategy, which underestimates the volatility in that regime, takes a very large short position in the risky asset. This hurts the out-of-sample performance of the strategy relative to the volatility timing strategy, probably because the negative expected return in those states is not precisely estimated.

If there are trading costs in the model, then λ will be switching through time between high and low transaction cost regimes. If an investor does not time the switches in λ, then the investor uses an unconditional average of λ_{avg} which is estimated from running the following regression in the liquid subset:

$$IS_i = \lambda_{\text{avg}} Q_i + \varepsilon_i$$

where Q_i is the dollar size of the order. We estimated λ_{avg} to be 0.766×10^{-10} which is between λ_1 and λ_2, as expected.

Now we consider combined timing strategies: Timing σ and μ, timing σ and λ or timing μ and λ. In all three timing strategies, the left-out parameter is set to its unconditional average. We consider the comparison across these policies in two different assumptions of γ: high risk-aversion and low risk-aversion.[23] Fig. A.3 compares these three policies in the presence of transaction costs in the high risk-aversion case. We observe that the top performing policy times σ and λ and the worst performing policy times σ and μ.

Figure A.8 compares these three policies in the low risk-aversion case. We again observe that the worst performing policy times σ and μ but the underperformance is economically smaller. This underscores that the benefits from timing volatility and transaction costs become more important when the size of the portfolio is large.

6. Conclusion

In this paper, we develop a closed-form solution for the dynamic asset allocation when expected returns, covariances, and price impact parameters follow a multi-state regime-switching model, and when the investor has a mean-variance objective function. We derive an optimal trading rule which is characterized by a set of aim portfolios and trading speed vectors. Specifically, the aim portfolio is a weighted average of the conditional Markowitz portfolios in all potential future states. The weight on each conditional Markowitz portfolio depends on the likelihood of transitioning to that state, the state's persistence, the risk, and transaction costs faced in that state compared to the current one. Similarly, the optimal trading speed is a function of the relative magnitude of the transaction costs in various states and their transition probabilities. One of the significant implications of our model is that the optimal portfolio can deviate substantially from the conditional Markowitz portfolio in anticipation of possible future shifts in relative risk and/or transaction costs.

We show that the model is equally tractable when either price changes or returns follow a regime-switching model. The latter aligns better with the empirical dynamics of asset returns. We utilize this framework to optimally time the broad value-weighted market portfolio, accounting for time-varying expected returns, volatility, and transaction costs. We use a large proprietary data set on institutional trading costs to estimate the price impact parameters. We find that trading costs vary significantly across regimes and tend to be higher as market volatility increases.

We test our trading strategy both in-sample and out-of-sample and find that there are substantial benefits to the use of our approach. For the out-of-sample test, the state probabilities are estimated using only data in the information set of an agent on the day preceding the trading date. We compare the performance of our optimal dynamic strategy to various benchmarks: a constant-dollar investment in the risky asset, a buy-and-hold portfolio, and a myopic policy with optimal trading speeds borrowed from the optimal solution. Our dynamic strategy outperforms all of these alternatives significantly. Out-of-sample, the benefits of timing volatility and transaction costs dominate those of timing expected returns, especially when assets under management are sizable.

[23] Note that in the absence of trading costs, changing the coefficient of risk-aversion would not matter, as the wealth values will just be scaled by the ratio of the risk-aversion parameters.

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Figures and Tables

Liquidity regimes and optimal dynamic asset allocation

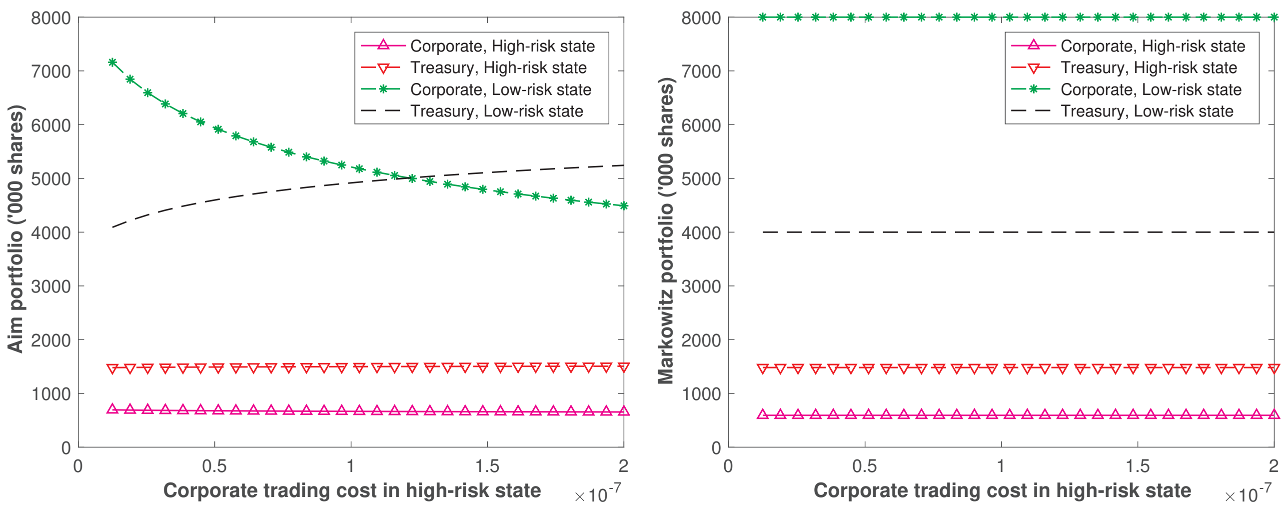

Fig. 1. Aim portfolio for the corporate bond example. Left panel plots the shares invested in the Corporate and Treasury in the aim portfolio in the high- and low-volatility states as the Corporate trading cost varies in the high-risk state. The formula for the aim portfolio is stated in Theorem 1. The right panel plots the conditional Markowitz portfolio in each state for direct comparison. All parameter values are provided in the left panel of Table 1.

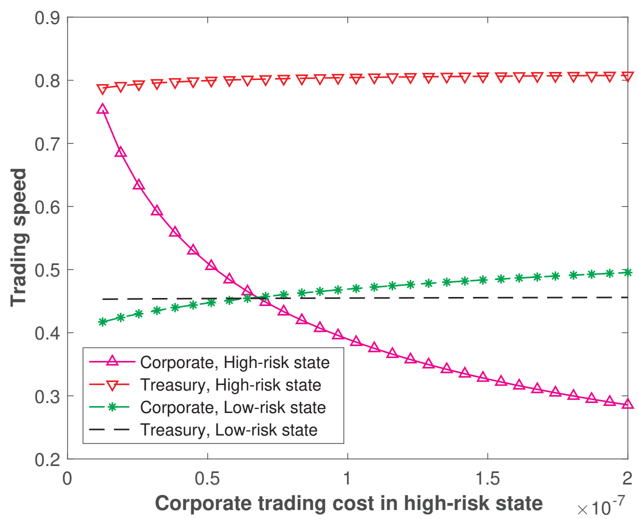

Fig. 2. Trading speed for corporate bond example. This plot gives the diagonal values of the trading speed matrix in each state ($\tau_s = \Lambda_s^{-1} Q_s$) as a function of Corporate trading cost in the high-risk state. The formula for the trading speed matrix is derived in Theorem 1. All parameter values are provided in the left panel of Table 1.

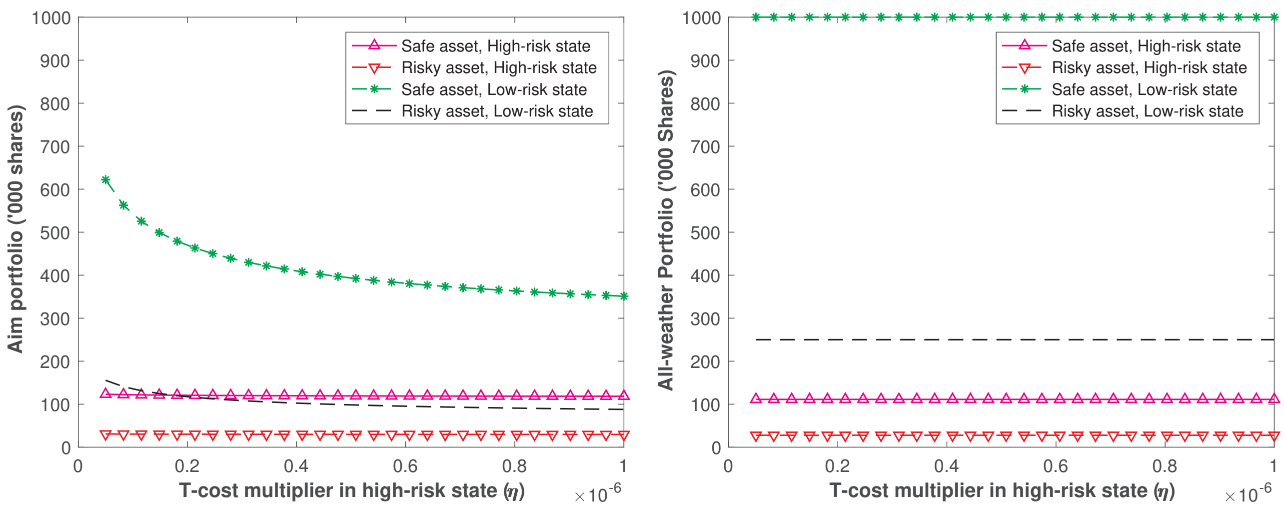

Fig. 3. Aim portfolio for risk-parity example. We plot the number of shares of the safe and risky asset in the aim portfolio for each volatility state, as a function of the transaction cost (T-cost) multiplier in the high-risk state, η. Note that $\Lambda_H = \eta\Sigma_H$. The formula for the aim portfolio is stated in Theorem 1. The right panel plots the conditional Markowitz portfolio in each state for direct comparison. All parameter values are provided in the right panel of Table 1.

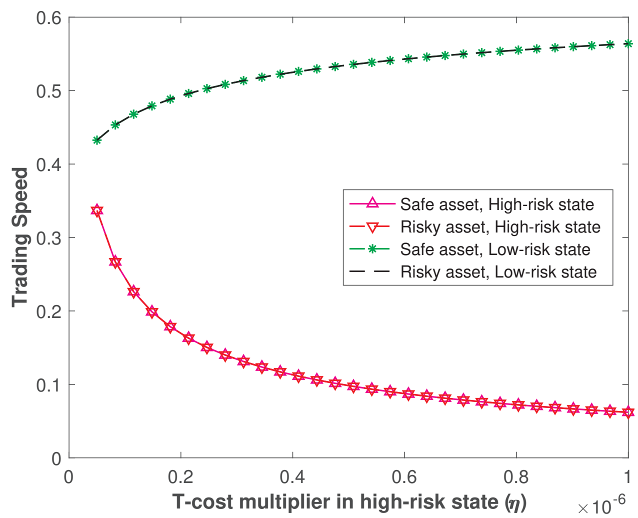

Fig. 4. Trading Speeds for the risk-parity experiment. This plot gives the diagonal values of the trading speed matrix in each state ($\tau_s = \Lambda_s^{-1} Q_s$) as a function of the transaction cost multiplier in the high-risk state, η. Note that $\Lambda_H = \eta \Sigma_H$. The formula for the trading speed matrix is derived in Theorem 1. All parameter values are provided in the right panel of Table 1.

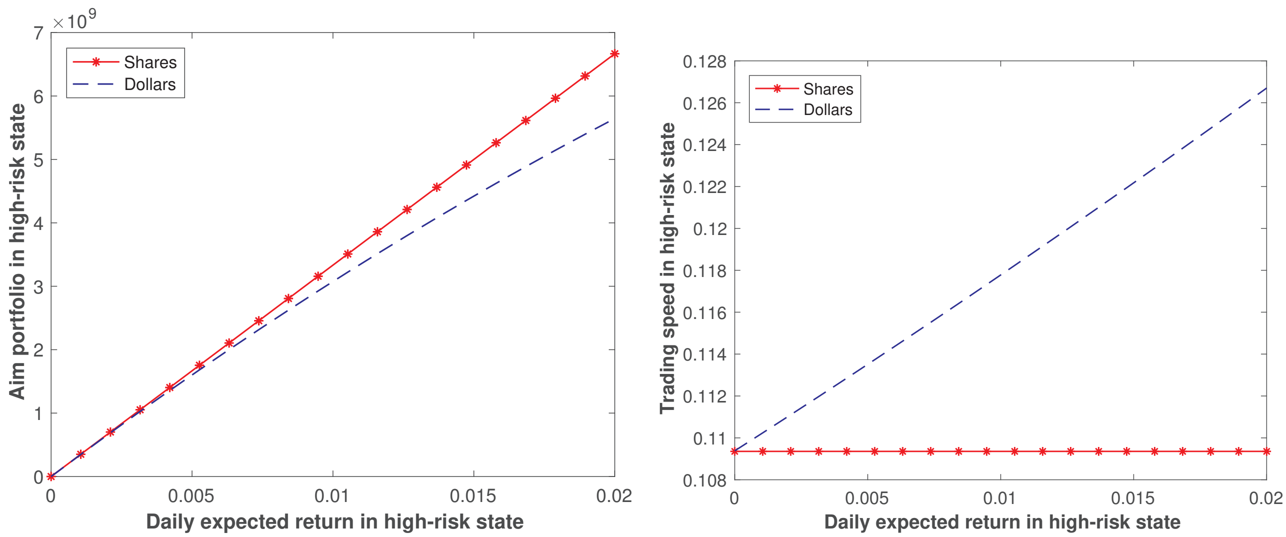

Fig. 5. Aim portfolios and trading speeds in models set up in shares and dollars. This figure compares the aim portfolios and trading speeds in two-state models set up in shares and dollars. The initial asset price is set to \$1 so that the models are directly comparable, i.e., expected returns and expected price changes are the same. Parameter values are given in Table 2. We report the aim portfolios and trading speeds in the high-risk state as a function of the expected return in that state.

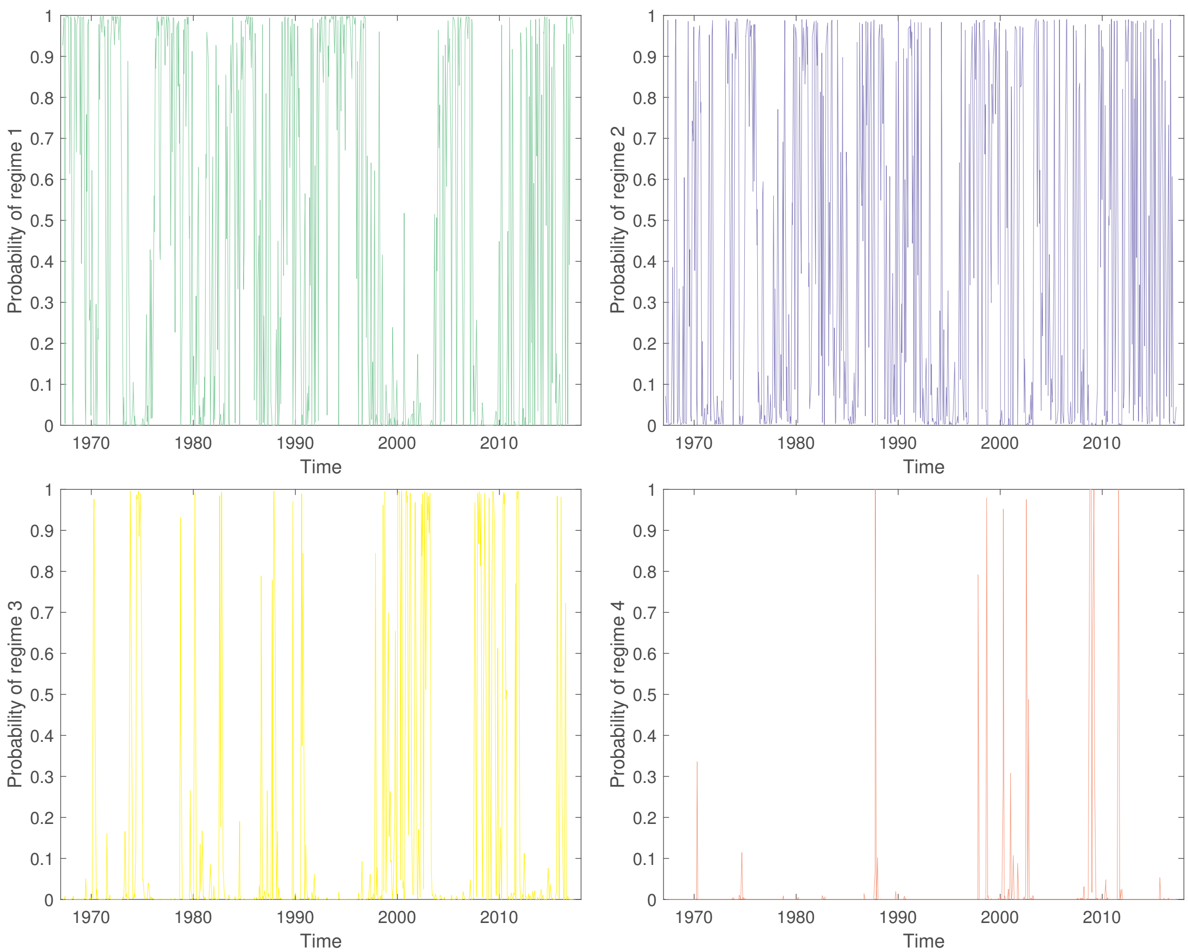

Fig. 6. Estimated state probabilities. We use daily value-weighted market returns of all firms in CRSP data set from 1967-03-13 to 2017-03-31 to estimate a four-state regime-switching model. The data are downloaded from Ken French's data library. The four plots illustrate the probability that the trading day t is in regime i conditional on the full return sample.

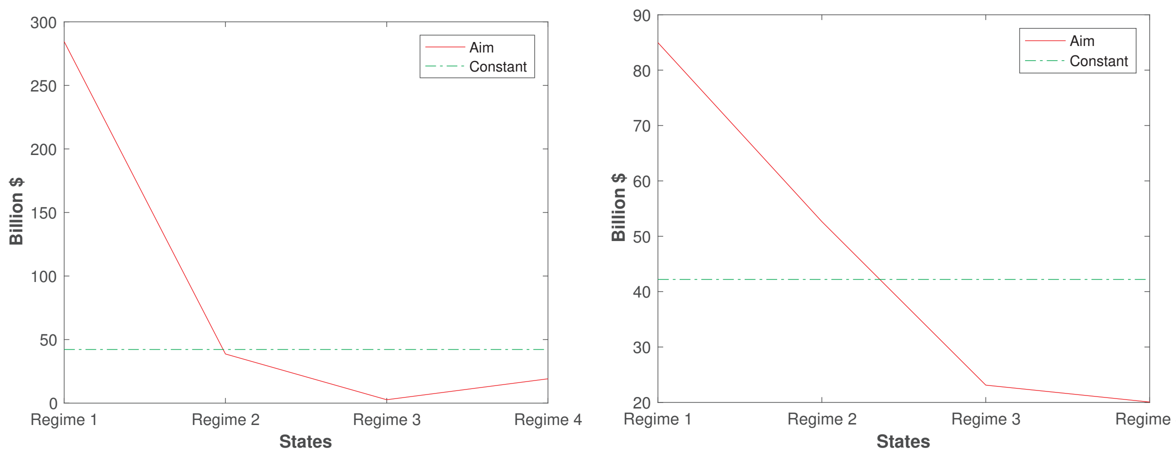

Fig. 7. Unconditional Markowitz portfolio and aim portfolios with and without trading costs. In the left panel, trading costs are assumed to be zero, thus the aim portfolio is equal to the conditional Markowitz portfolio. We also plot the unconditional Markowitz portfolio which we label as the "Constant" portfolio. In the right panel, trading costs are positive and we use trading cost multipliers as set in Section 5.2. The coefficient of risk-aversion is given by $\gamma = 1 \times 10^{-10}$ (can be thought of as corresponding to a relative risk-aversion of 1 for an agent with \$10 billion dollars under management).

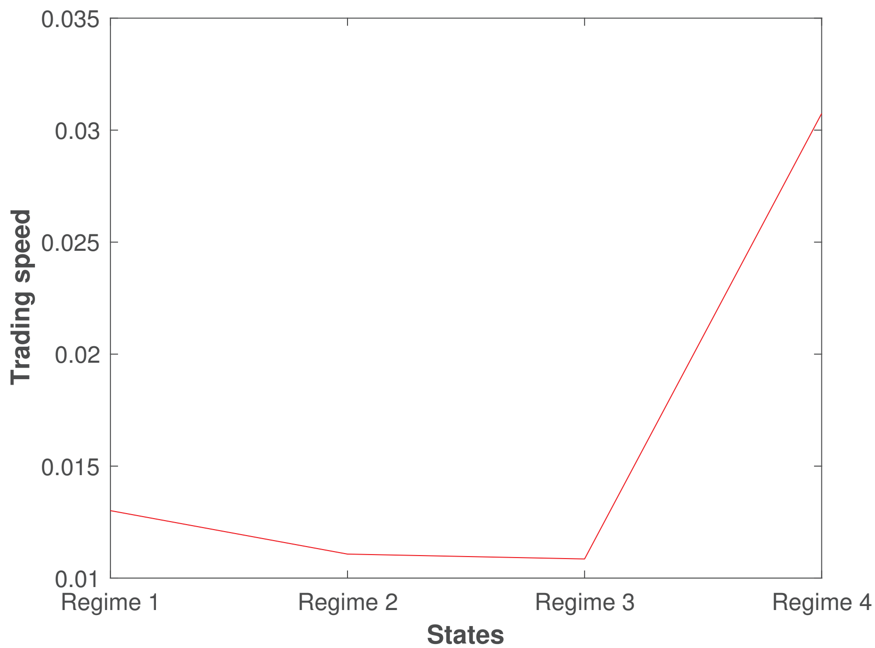

Fig. 8. Trading speed across different regimes. Trading costs are set according to Table 5 using executions on very large-cap stocks. Coefficient of risk-aversion is given by $\gamma = 1 \times 10^{-10}$ (can be thought of as corresponding to a relative risk-aversion of 1 for an agent with \$10 billion dollars under management).

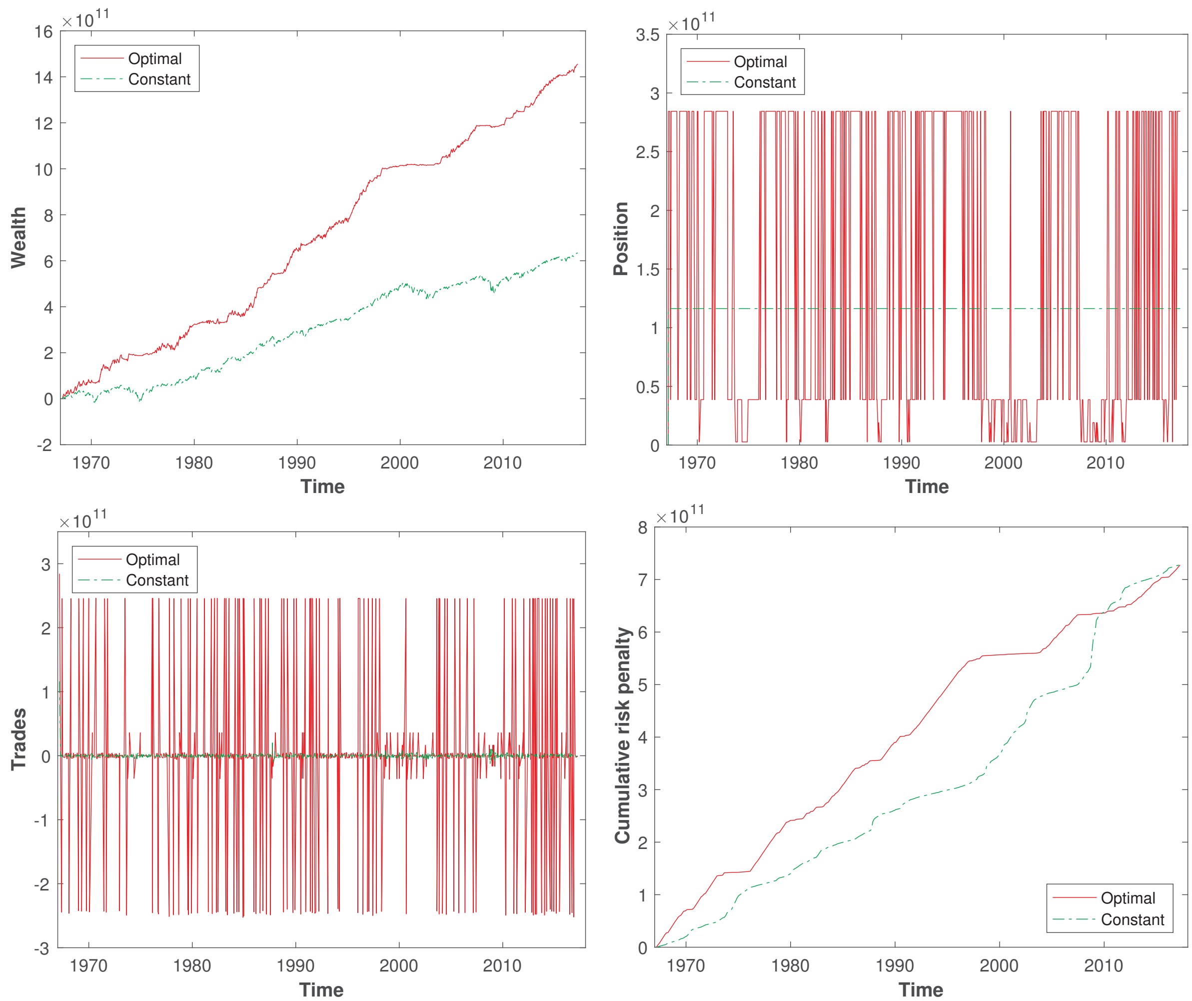

Fig. 9. This figure compares the in-sample performance of the optimal policy with a constant-dollar portfolio in the absence of trading costs. Both strategies start from zero-wealth. Constant portfolio is equal to the scaled unconditional Markowitz portfolio so that the policy has the same risk exposure as the optimal policy (as shown in the bottom-right panel). Top-left panel shows the cumulative wealth of each policy which is the main comparison metric. Top-right panel shows each strategy's dollar position in the market fund. Bottom-left panel illustrates the change in position due to the rebalancing of the strategy.

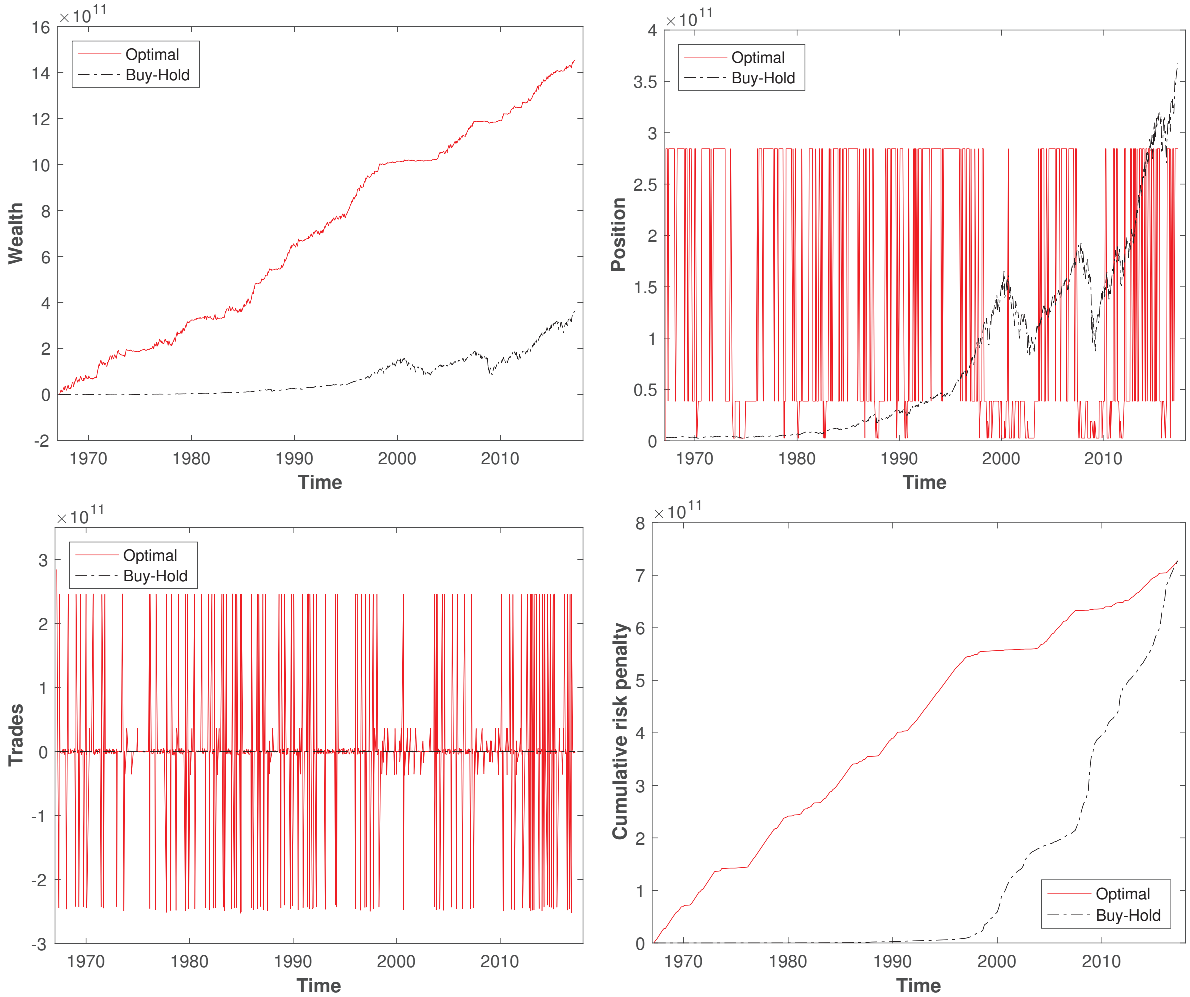

Fig. 10. This figure compares the in-sample performance of the optimal policy with a buy-and-hold portfolio in the absence of trading costs. Both strategies start from zero-wealth. Buy-and-hold portfolio invests x_0 dollars into the market fund at the beginning of the horizon. We scale x_0 so that the policy has the same risk exposure as the optimal policy (as shown in the bottom-right panel). Top-left panel shows the cumulative wealth of each policy which is the main comparison metric. Top-right panel shows each strategy's dollar position in the market fund. Bottom-left panel illustrates the change in position due to the rebalancing of the strategy.

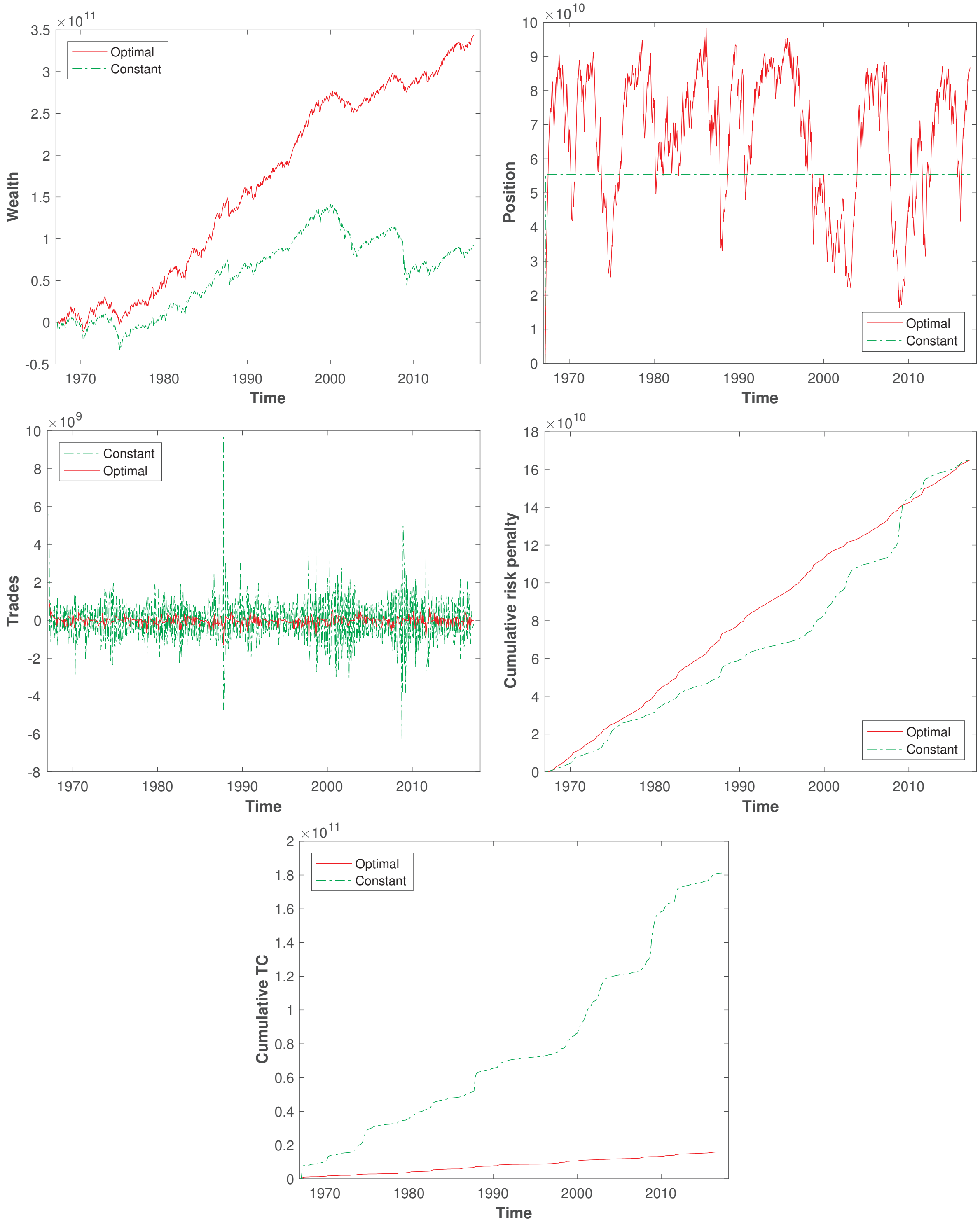

Fig. 11. This figure compares the in-sample performance of the optimal policy with a constant-dollar portfolio in the presence of trading costs (TC). Both strategies start from zero-wealth. Constant portfolio is equal to the scaled unconditional Markowitz portfolio so that the policy has the same risk exposure as the optimal policy (as shown in the center-right panel). Top-left panel shows the cumulative wealth of each policy which is the main comparison metric. Top-right panel shows each strategy's dollar position in the market fund. Center-left panel illustrates the change in position due to the rebalancing of the strategy and the bottom panel illustrates the cumulative cost of these trades.

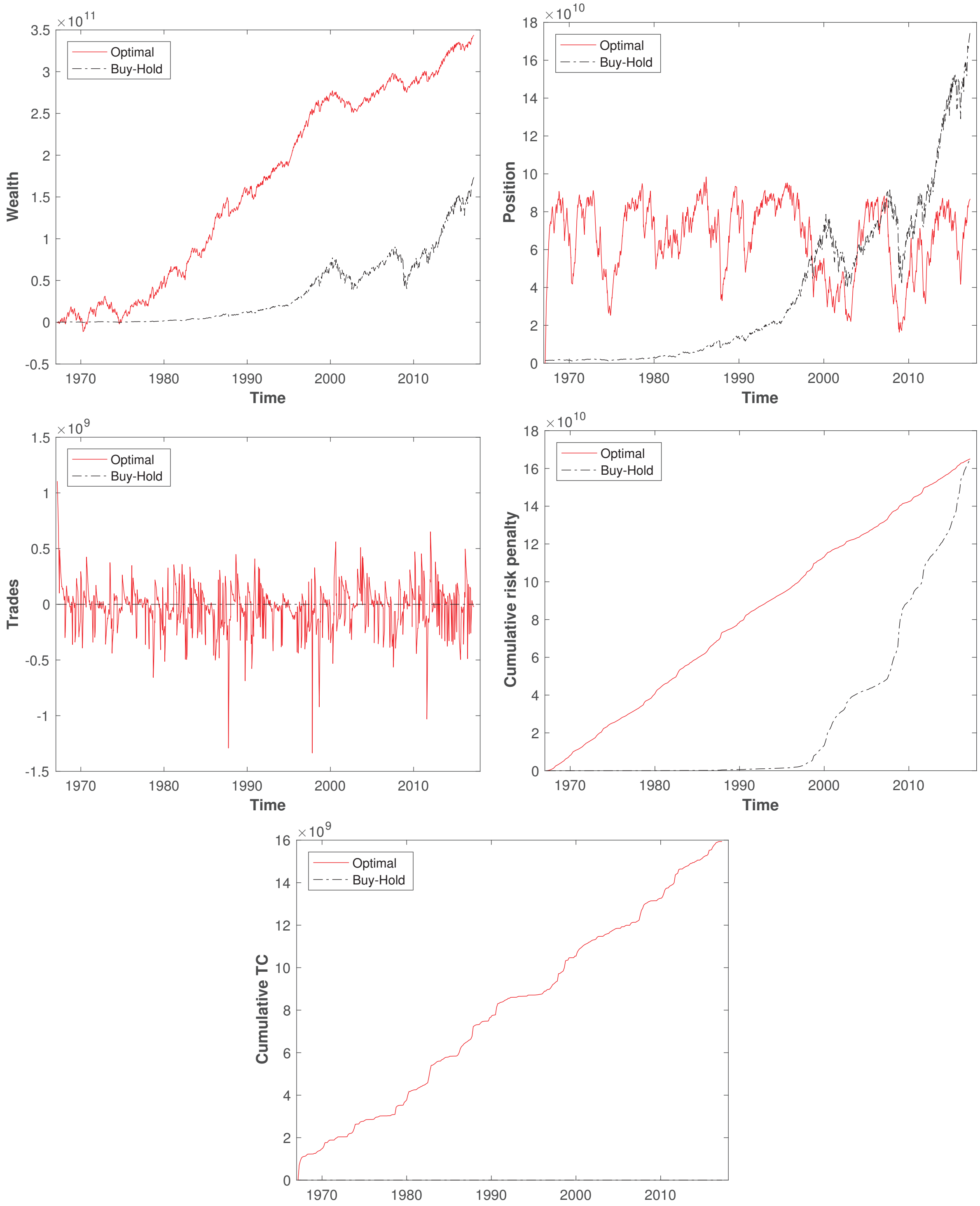

Fig. 12. This figure compares the in-sample performance of the optimal policy with a buy-and-hold portfolio in the presence of trading costs. Both strategies start from zero-wealth. Buy-and-hold portfolio invests x_0 dollars into the market fund at the beginning of the horizon. We scale x_0 so that the policy has the same risk exposure as the optimal policy (as shown in the center-right panel). Top-left panel shows the cumulative wealth of each policy which is the main comparison metric. Top-right panel shows each strategy's dollar position in the market fund. Center-left panel illustrates the change in position due to the rebalancing of the strategy and the bottom panel illustrates the cumulative cost of these trades.

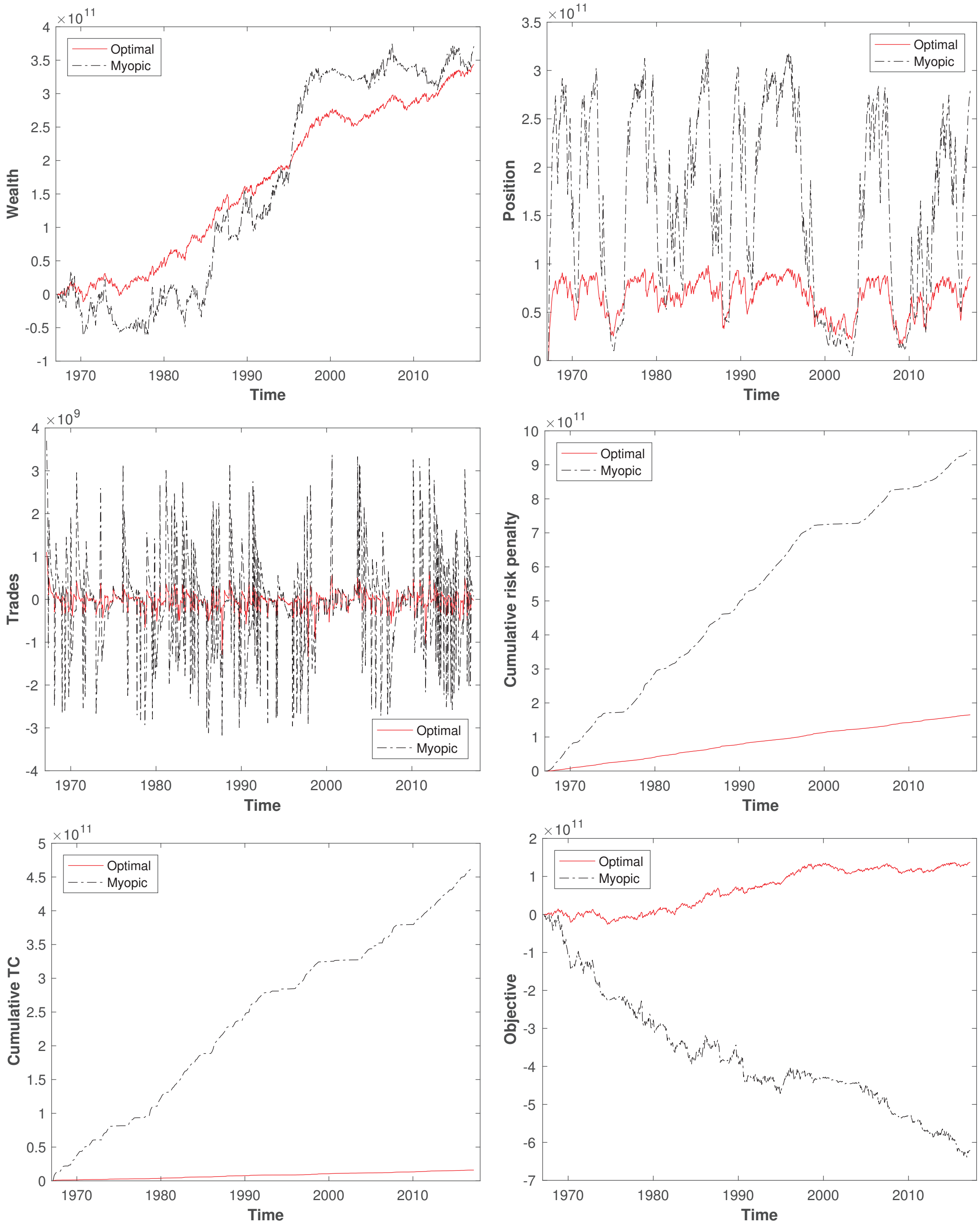

Fig. 13. This figure compares the in-sample performance of the optimal policy with a myopic policy in the presence of trading costs. Both strategies start from zero-wealth. We adjust the myopic policy so that it has the same trading speed as the optimal policy. Top-left panel shows the cumulative wealth of each policy. Top-right panel shows each strategy's dollar position in the market fund. Center-left panel illustrates the change in position due to the rebalancing of the strategy and the bottom-left panel illustrates the cumulative cost of these trades. Bottom-right panel shows the realized objective value of each strategy, which is the main comparison metric, by subtracting the risk penalty from the wealth generated.

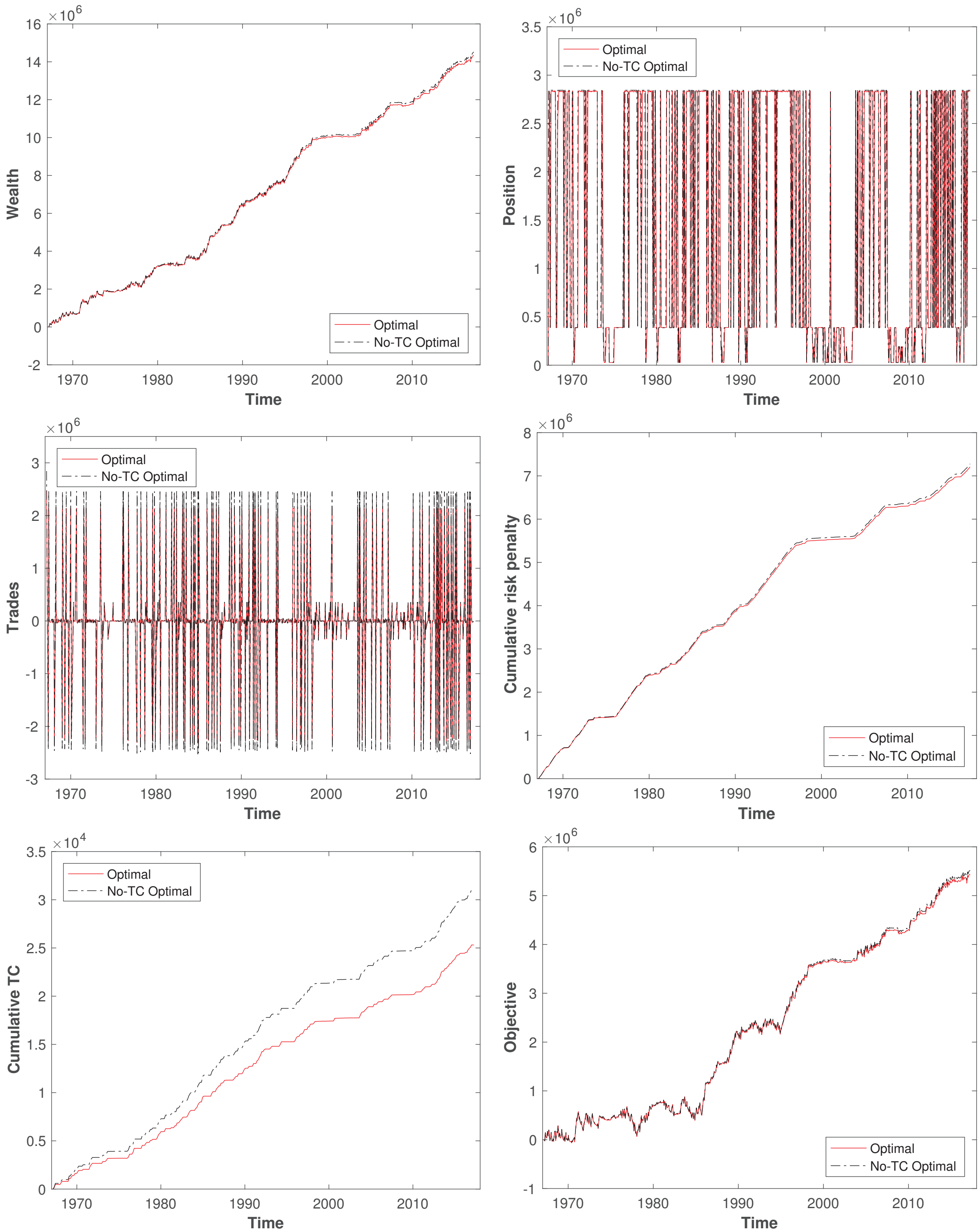

Fig. 14. This figure compares the in-sample performance of the optimal policy with a conditional Markowitz portfolio for a small investor ($\gamma = 1 \times 10^{-5}$) in the presence of trading costs. Recall that conditional Markowitz portfolio is optimal in the absence of trading costs. Both strategies start from zero-wealth. Top-left panel shows the cumulative wealth of each policy. Top-right panel shows each strategy's dollar position in the market fund. Center-left panel illustrates the change in position due to the rebalancing of the strategy and the bottom-left panel illustrates the cumulative cost of these trades. Bottom-right panel shows the realized objective value of each strategy, which is the main comparison metric, by subtracting the risk penalty from the wealth generated.

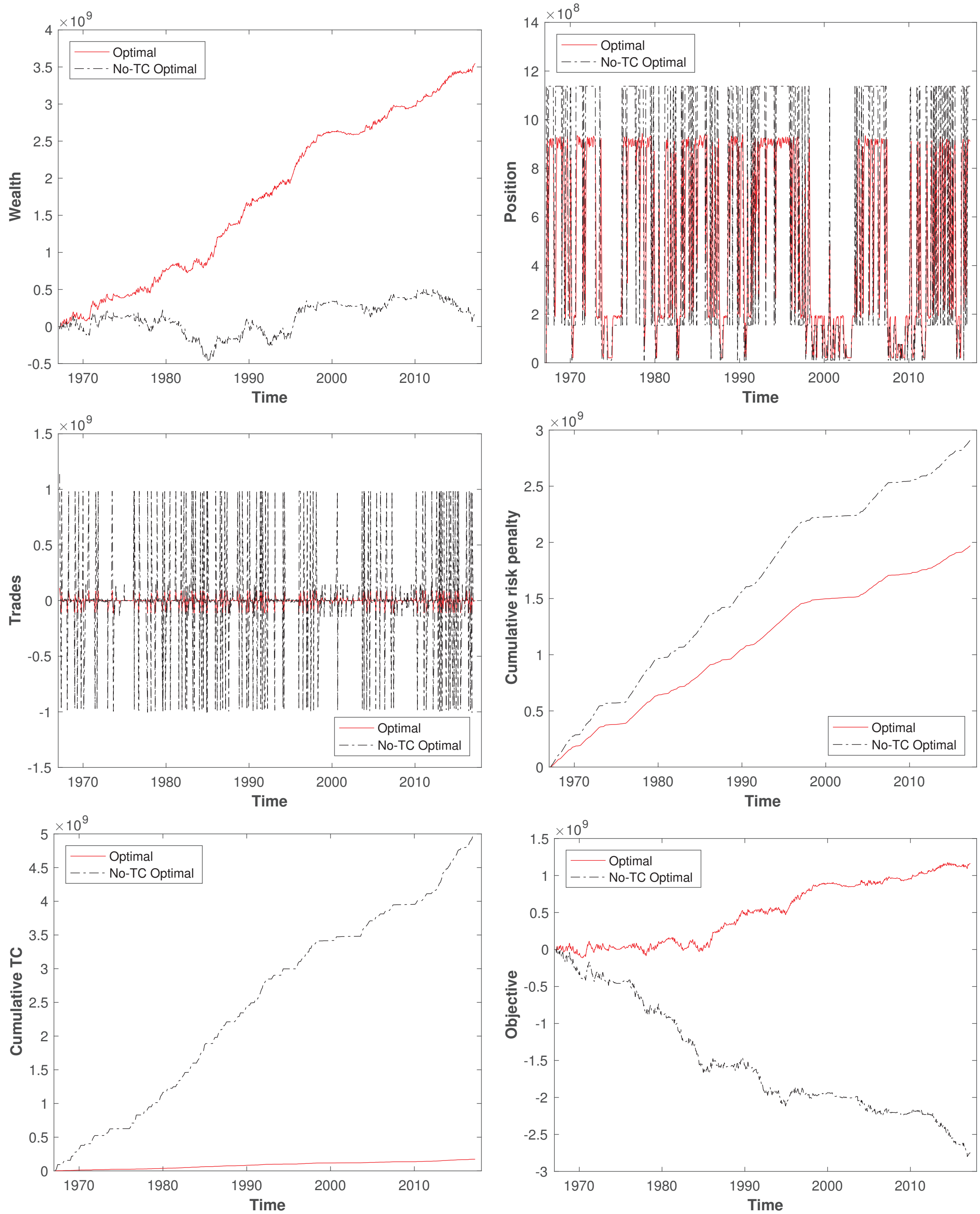

Fig. 15. This figure compares the in-sample performance of the optimal policy with a conditional Markowitz portfolio for a medium-size investor ($\gamma = 2.5 \times 10^{-8}$) in the presence of trading costs. Recall that conditional Markowitz portfolio is optimal in the absence of trading costs. Both strategies start from zero-wealth. Top-left panel shows the cumulative wealth of each policy. Top-right panel shows each strategy's dollar position in the market fund. Center-left panel illustrates the change in position due to the rebalancing of the strategy and the bottom-left panel illustrates the cumulative cost of these trades. Bottom-right panel shows the realized objective value of each strategy, which is the main comparison metric, by subtracting the risk penalty from the wealth generated.

Table 1
Parameter values for corporate bond and risk-parity examples.
This table reports the parameter values used in the numerical experiments described in Sections 3.1 and 3.2. Trading is daily, and L denotes the low-risk state and H denotes the high-risk state. Reported values of expected price-change, μ, and price-change covariance matrix, Σ, are annualized for each state. Λ denotes the price impact matrix in each state. γ denotes the coefficient of risk-aversion and ρ is the discount rate. π_{LL} (π_{HH}) denotes the transition probability of remaining in the low-risk (high-risk) state. The initial prices for all assets are \$100. In the left panel, the first (second) row in μ, Σ, and Λ corresponds to the Corporate (Treasury). In the right panel, the first (second) row in μ, Σ, and Λ corresponds to the "safe" ("risky") asset.

Corporate bond example		Risk-parity example	
Parameter	Value	Parameter	Value
γ	10^{-8}	γ	10^{-8}
ρ	0.9996	ρ	0.9996
π_{LL}	0.95	π_{LL}	0.95
π_{HH}	0.9	π_{HH}	0.9
μ_L	$\begin{bmatrix} 10 \\ 8 \end{bmatrix}$	μ_L	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
μ_H	$\begin{bmatrix} 12 \\ 16 \end{bmatrix}$	μ_H	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
Σ_L	$\begin{bmatrix} 100 & 50 \\ 50 & 100 \end{bmatrix}$	Σ_L	$\begin{bmatrix} 100 & 0 \\ 0 & 900 \end{bmatrix}$
Σ_H	$\begin{bmatrix} 900 & 450 \\ 450 & 900 \end{bmatrix}$	Σ_H	$\begin{bmatrix} 400 & 0 \\ 0 & 3600 \end{bmatrix}$
Λ_L	$\begin{bmatrix} 1.25 \times 10^{-8} & 0 \\ 0 & 10^{-8} \end{bmatrix}$	Λ_L	$5 \times 10^{-8}\Sigma_L$
Λ_H	$\begin{bmatrix} \text{Variable} & 0 \\ 0 & 10^{-8} \end{bmatrix}$	Λ_H	(Variable) $\eta\Sigma_H$

Table 2
Parameter values for the price-change and return models compared in Section 4.4.
This table presents the parameter values used in the comparison between price-change and return models in Section 4.4, the results of which are plotted in Fig. 5. The initial price for the risky asset in both settings in \$1. Trading is daily and L denotes the low-risk state and H denotes the high-risk state. The expected price-change/return (μ) and price-change/return variance (Σ) are daily. Λ denotes the price impact matrix in each state. γ denotes the coefficient of risk-aversion and ρ is the discount rate. π_{LL} (π_{HH}) denotes the transition probability of remaining in the low-risk (high-risk) state.

Parameter	Value	Parameter	Value
γ	5×10^{-8}	μ_H	(Variable)
ρ	0.9996	Σ_L	$\left[0.40 \times 10^{-4}\right]$
π_{LL}	0.98	Σ_H	$\left[3.33 \times 10^{-4}\right]$
π_{HH}	0.9	Λ_L	$\left[2 \times 10^{-10}\right]$
μ_L	$\left[0\right]$	Λ_H	$\left[3 \times 10^{-10}\right]$

Table 3
Parameter estimates for a four-state regime-switching model.

This table presents estimates of the model parameters presented in Eq. 15 using daily value-weighted CRSP market return data from 1967-03-13 to 2017-03-31. The data are downloaded from Ken French's data library. All parameter estimates are statistically significant at 1% level.

Parameter	Estimate	Parameter	Estimate
μ_1	0.0864%	σ_1	0.5512%
μ_2	0.0340%	σ_2	0.9372%
μ_3	0.0069%	σ_3	1.6032%
μ_4	0.2939%	σ_4	3.9178%
P_{11}	0.9804	P_{12}	0.0196
P_{13}	0.0000	P_{14}	0.0000
P_{21}	0.0250	P_{22}	0.9670
P_{23}	0.0080	P_{24}	0.0000
P_{31}	0.0000	P_{32}	0.0233
P_{33}	0.9693	P_{34}	0.0074
P_{41}	0.0016	P_{42}	0.0000
P_{43}	0.0635	P_{44}	0.9350

Table 4
Summary statistics for the main attributes in the execution data.

The execution data cover S&P 500 stocks between January 2011 and December 2012. Participation rate is equal to the ratio of the executed volume to total volume during the lifetime of the order. The volatility of the asset is estimated using the mid-quote prices. Order duration is expressed as a fraction of full trading day (i.e., 6.5 hours). Implementation shortfall (IS) is expressed in basis points (bps).

Statistic	Mean	Min	Pctl(25)	Median	Pctl(75)	Max
Order value ($ M)	0.967	0.001	0.094	0.396	1.102	158.300
Participation rate	0.061	0.00001	0.002	0.013	0.102	1.000
Volatility	0.014	0.0002	0.008	0.011	0.016	0.344
Order duration	0.384	0.013	0.041	0.153	0.851	1.000
IS (bps)	4.095	-1006.000	-16.440	4.075	25.080	996.600

Table 5
Transaction cost estimates in each regime for the four-state regime-switching model.

λ_n denotes the transaction cost multiplier in regime n. The second column reports the results from a liquid subset in which we only include executions from stocks within the top 10% in market capitalization. Estimated values are multiplied by 10^{10}. Standard errors are double-clustered at the stock and calendar day levels. * denotes $p < 0.1$; ** $p < 0.05$; and *** denotes $p < 0.01$.

	Dependent variable: IS	
	All stocks	*Liquid*
λ_1	1.688***	0.501**
	(0.459)	(0.217)
λ_2	1.725***	0.793***
	(0.195)	(0.189)
λ_3	3.037***	1.506***
	(0.418)	(0.352)
λ_4	2.274	0.935
	(1.927)	(1.329)

Table 6
Average liquidity proxies in each regime.

The second column reports the averages from a liquid subset in which we only include executions from stocks within the top 10% in market capitalization. Standard errors, in parentheses, are double-clustered at the stock and calendar day levels. *** denotes $p < 0.01$

	All stocks				Liquid			
	Spread (bps)	Volatility (%)	Turnover	Amihud ($\times 10^{-8}$)	Spread (bps)	Volatility (%)	Turnover	Amihud ($\times 10^{-8}$)
1	3.80***	1.11***	3.51***	1.37***	2.89***	1.01***	2.93***	0.47***
	(0.04)	(0.02)	(0.12)	(0.06)	(0.03)	(0.02)	(0.11)	(0.02)
2	3.95***	1.23***	3.58***	1.59***	2.98***	1.13***	3.02***	0.55***
	(0.04)	(0.02)	(0.13)	(0.06)	(0.03)	(0.02)	(0.10)	(0.02)
3	4.95***	1.92***	4.27***	1.97***	3.52***	1.77***	3.63***	0.67***
	(0.09)	(0.06)	(0.20)	(0.08)	(0.07)	(0.06)	(0.19)	(0.04)
4	5.62***	2.84***	7.37***	1.80***	3.97***	2.59***	6.42***	0.59***
	(0.39)	(0.28)	(2.07)	(0.34)	(0.27)	(0.25)	(1.07)	(0.12)

Table 7
Parameter estimates for the two-state regime-switching model.

This table presents estimates for the parameters governing the two-state regime-switching model based on using daily value-weighted CRSP market return data from 1926-07-01 to 2017-03-31. The data are downloaded from Ken French's data library. All parameter estimates are statistically significant at 1% level.

Parameter	Estimate	Parameter	Estimate
μ_1	0.0841%	σ_1	0.6110%
μ_2	-0.0955%	σ_2	1.8886%
P_{11}	0.9866	P_{12}	0.0134
P_{21}	0.0431	P_{22}	0.9569

Table 8
Transaction cost estimates for the two-state regime-switching model.

Transaction cost estimates for each regime are estimated for the two-state regime-switching model. λ_n denotes the transaction cost multiplier in regime n. The second column reports the results from a liquid subset in which we only include executions from stocks within the top 10% in market capitalization. Estimated values are multiplied by 10^{10}. Standard errors, in parentheses, are double-clustered at the stock and calendar day levels. *** denotes $p < 0.01$.

	Dependent variable: IS	
	All stocks	*Liquid*
λ_1	1.772***	0.579***
	(0.255)	(0.166)
λ_2	2.299***	1.254***
	(0.311)	(0.335)